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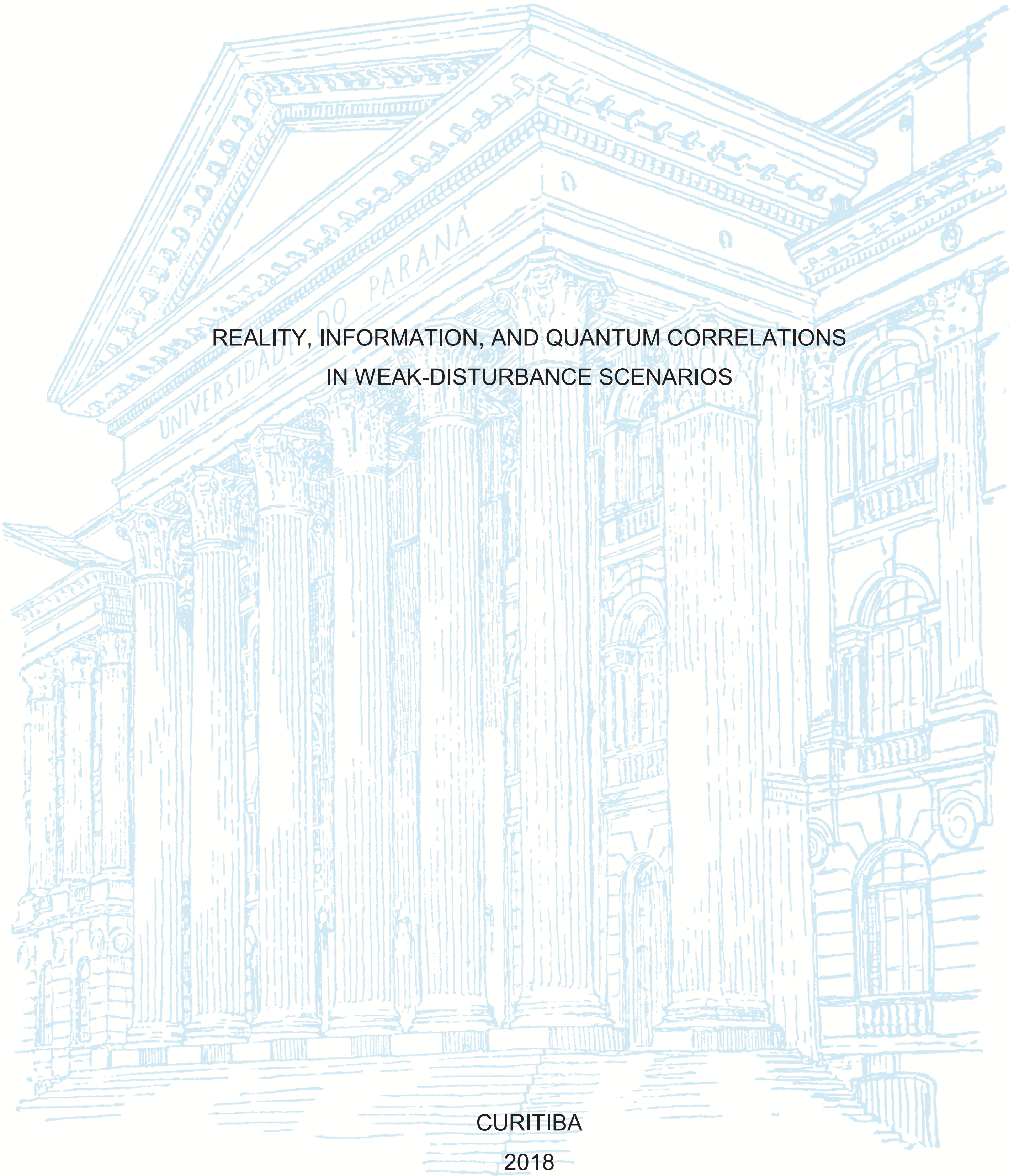

REALITY, INFORMATION, AND QUANTUM CORRELATIONS IN WEAK-DISTURBANCE SCENARIOS

CURITIBA

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Universidade Federal do Paraná

PEDRO RUAS DIEGUEZ

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Tese apresentada ao Programa de Pós-graduação em Física do Setor de Ciências Exatas da Universidade Federal do Paraná, como parte dos requisitos necessários à obtenção do grau de Doutor em Física.

Orientador: Prof. Dr. Renato Moreira Angelo

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Os membros da Banca Examinadora designada pelo Colegiado do Programa de Pós-Graduação em FÍSICA da Universidade Federal do Paraná foram convocados para realizar a arguição da tese de Doutorado de **PEDRO RUAS DIEGUEZ** intitulada: **"Reality, information, and quantum correlations in weak-disturbance scenarios"**, após terem inquirido o aluno e realizado a avaliação do trabalho, são de parecer pela sua APROVAÇÃO no rito de defesa.

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RESUMO

Recentemente, foi apresentada uma medida que permite quantificar o grau de realidade de um dado observável para uma determinada preparação. Nesta tese, empregamos este quantificador para estabelecer, em bases formais, relações entre os conceitos de realidade física, informação e correlações quânticas no regime de medições fracas. Como primeiro resultado, introduzimos objetos matemáticos que unificam as medições fracas e projetivas com os resultados das medições sendo revelados e também para o caso de medições não-reveladas. Com isso, estudamos cenários mostrando que uma medida não-revelada de intensidade arbitrária de um dado observável geralmente leva a um aumento de sua realidade e também de seus observáveis incompatíveis. Notavelmente, derivamos uma relação de complementaridade conectando a quantidade de informação associada ao aparato com o grau de irrealidade do observável monitorado. Também apontamos alguns mecanismos pelos quais a irrealidade de um observável pode ser gerada. Especificamente para estados puros, mostramos que o emaranhamento com o aparato determina precisamente a quantidade pela qual a realidade dos observáveis monitorados aumenta. Na tentativa de estudar relações mais gerais entre os conceitos de realidade e correlações quânticas, olhamos para uma medida conhecida como discórdia quântica, para estudar o papel das medições de intensidade arbitrária para a definição das correlações quânticas. Originalmente introduzida como a diferença entre duas formas possíveis de informação mútua quântica, a discórdia quântica tem posteriormente mostrado admitir uma formulação de acordo com a qual ela mede a distância entre o estado sob escrutínio e o estado mais próximo medido projetivamente (não discordante). Recentemente, foi demonstrado que a discórdia quântica resulta em valores maiores quando as medições projetivas são substituídas por medições fracas. Isso parece paradoxal, uma vez que medições mais fracas deveriam implicar um distúrbio mais fraco e, portanto, uma distância menor. Nesta tese, resolvemos esse conflito apresentando um quantificador e uma interpretação subjacente para o que chamamos de discórdia quântica fraca. Como subproduto, introduzimos a noção de discórdia quântica fraca simétrica. Finalmente, usando as ferramentas mencionadas anteriormente, construímos uma imagem consistente para abordar vários problemas fundamentais, como o problema de medição, o paradoxo de Everett, o paradoxo do amigo de Wigner e a dualidade onda-partícula.

Palavras-chave: Realidade. Informação. Medições. Entropia.

ABSTRACT

Recently, a measure has been put forward which allows for the quantification of the degree of reality of an observable for a given preparation. In this thesis we employ this quantifier to establish, on formal grounds, relations among the concepts of physical reality, information, and quantum correlations in the weak disturbance measurements regime. As a first result we introduce mathematical objects that unify weak and projective measurements for both revealed and unrevealed measurement results. With that, we study scenarios showing that an arbitrary-intensity unrevealed measurement of a given observable generally leads to an increase of its reality and also of its incompatible observables. Remarkably, we derive a complementarity relation connecting an amount of information associated with the apparatus with the degree of irreality of the monitored observable. We also point out some mechanisms whereby the irreality of an observable can be generated. Specifically for pure states, we show that the entanglement with the apparatus precisely determines the amount by which the reality of the monitored observable increases. In an attempt to study more general relations between the concepts of reality and quantum correlations, we look at a measure known as quantum discord, to study the role of measurements of arbitrary intensity for the definition of quantum correlations. Originally introduced as the difference between two possible forms of quantum mutual information, quantum discord has posteriorly been shown to admit a formulation according to which it measures a distance between the state under scrutiny and the closest projectively measured (non-discordant) state. Recently, it has been shown that quantum discord results in higher values when projective measurements are substituted by weak measurements. This sounds paradoxical since weaker measurements should imply weaker disturbance and, thus, a smaller distance. In this thesis we solve this puzzle by presenting a quantifier and an underlying interpretation for what we call weak quantum discord. As a by-product, we introduce the notion of symmetrical weak quantum discord. Finally, using the aforementioned tools, we construct a consistent picture to address several foundational problems such as, the measurement problem, Everett's paradox, Wigner's friend paradox, and the wave-particle duality.

Key-words: Reality. Information. Measurements. Entropy.

LIST OF FIGURES

SUMMARY

1 INTRODUCTION

Experiments carried out to test the celebrated result developed by Bell [1] show us that the image of reality that we have of our everyday experience can not be a true representation of nature [2, 3, 4]. Equipped with the superposition principle, quantum mechanics (QM) teaches us that this view cannot be generally maintained, in fact, it has repeatedly been shown by experiments with isolated microscopic systems that the classical notion of physical reality is objectionable. We can say that this image is due in large part to the great success of the Newtonian description of nature, which as we know, is based on a deterministic theory, deeply interpreted in the words of Laplace [5]:

> "We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes."

Supported by the theory of QM, such intellect baptized as the Laplace demon, even though considering that it has the knowledge of all the forces that takes place in nature, it would not be able to accurately determine both the momentum and the position of the smallest physical systems, since they must obey the uncertainty principle. QM has satisfactorily explained a wide range of phenomena which until the 19th century were considered enigmatic, such as the existence of spectral lines, the stability of matter, the chemical nature of elements, ferromagnetism, superconductivity, etc [6, 7, 8]. However, since its inception in the 1920s, QM has attracted a great attention due to the significant changes in our understanding of what constitutes a scientific theory to describe nature and, in particular, to the new role assigned to the measurement process. Unlike other physical theories, the principle of superposition together with the Born rule implied that the temporal evolution of a closed physical system is no longer deterministic for a given initial condition. Then, the question of observation in QM gained a completely different status, challenging our philosophical conceptions about the nature of physical reality and the role associated with the observer. With the rapidly growing field of the quantum information theory and with the need of implementation of the so-called quantum technology, like quantum computation and quantum cryptography, several interpretations were created to explain counter-intuitive quantum phenomena. This recent wave of foundational studies is not surprising, as Khrennikov writes in [9], given that the success of quantum information theory is shadowed by the recognition that the basic foundational problems of QM have not yet been solved, even considering one hundred years of tremendous efforts to do that.

One of the main interpretative problems of QM arises due to a supposed conflict between two quantum processes. On the one hand, the temporal evolution of a closed quantum system is given by a unitary and therefore reversible and deterministic operator (as is the case of classical mechanics), as described by the Schrödinger equation. Like in classical mechanics, backwards time evolution can be mathematically used to predict the past of the system, that is, the action of the unitary operator in a quantum state is temporally symmetrical. However, when performing a measurement on a state that is not an eigenstate of the observable which is being measured, it occurs (in some interpretations [10, 11, 12]) the phenomenon known as the collapse of the wave-function, which is an intrinsically random process and, therefore, an irreversible projection onto one of the eigenstates of the observable that is being measured. In this scenario, the following question naturally arises: What is the nature of the measurement process in QM that makes it completely different from any other unitary interaction between quantum systems? This is the basic question of the controversial *Measurement problem* in QM [11, 13, 14, 15, 16, 17].

It is controversial in a sense that many researchers adopt drastically different views about the new role attributed to the measurement process. It is interesting to see how answers can vary in this respect [18]. For some, the measurement problem is only a "pseudo problem"[11], in a way that the postulate of reduction is an inherent aspect of the theory and consistent with unitary evolution. Others attribute to the measurement process a prominent role and even believe that without a solution, it would not be possible to understand the true nature belonging to the domains of quantum theory. As a consequence, a large spectrum of different philosophical views (see [12] for details) was adopted by physicists throughout the 20th century. The famous Copenhagen interpretation avoids the problem assuming that the measuring device is a classical system in certain sense and, as such, should be seen as an element external to the quantum description. The positivistic movement that was strongly founded on an empiricist approach [19, 20], defending a "silence" on what can not be observed, strongly influenced the founding fathers of QM. However, von Neumann proposed a model that deals with the measurement process as a quantum interaction [21] between two quantum systems, in such a way that we attribute a state vector to the measuring system, where the total system (measuring device and system to be measured) is considered closed and evolves according to Schrödinger equation. Initially recognized by von Neumann himself and reiterated by Fine in the 1970s, the linearity of QM seemed to put an end in attempts to solve the measurement problem from an interpretation in which the collapse of the wave function is an ontological component of the theory. This can be understood by looking at what is known as the insolubility theorem of the measurement problem as per the von Neumann model [22] (see [23], for a more pedagogical version of this theorem).

In this context, the decoherence program [15, 24, 25, 26, 27, 28], initially proposed by Zeh, begins to consider the influence of an external environment to the quantum system as an essential ingredient for the measurement process, arguing convincingly (for many) that, as the measuring apparatus is a macroscopic system, it would be impossible to treat it without taking into account the effects caused by the surrounding environment. The correlations established between the environment E, and the apparatus A imply an open dynamics for the joint system $S + A$ which, without quantum coherences, allows for a statistical solution to the measurement problem. The decoherence program gained considerable visibility in the 1980s with the work of Zurek, [16, 17], who made significant contributions to the foundations of quantum theory and to the measurement problem by showing how an external environment could be used to explain the irreversibility observed as the result of a measurement process in QM.

Since the result of a measurement in QM is an intrinsically random and irreversible process, it could be responsible for the apparent temporal asymmetry of nature (the so-called "arrow of time"), despite the fact that the fundamental laws of physics are temporally symmetrical. Admitting that this temporal asymmetry in QM is not an intrinsic aspect of the theory but rather a problem in the choice of temporal boundary conditions given to quantum states, Aharonov and collaborators developed in 1964 the Two state vector formalism (TSVF) [29], with the aim of removing such temporal asymmetry. This is a reformulation of the usual QM in terms of pre- and post-selected ensembles [30, 31]. These ideas culminated in asking if it would be possible to make a measurement of a quantum state without inducing a collapse to one of the eigenstates of the measured observable. To answer this question, Aharonov, Albert, and Vaidman in 1988 developed the concepts of weak measurement and weak values. The weak measurement can be understood as an infinitesimal interaction between the system to be measured and the apparatus, with the intention of not substantially changing the state to be measured. Based on the TSVF it is possible to develop an approach closer to the Copenhagen view, in the sense that both unitary and the reduction postulate are ontological elements although been *complementarity* in Bohr's sense [32]. This is not the only possible interpretation for this formalism. In Everett's interpretation [33], also known as Many Worlds interpretation [34], the collapse of the wave function does not even occur, for all branches of a quantum superposition are "elements of physical reality" even considering that this alternative realities exists in mutually exclusive worlds.

Regardless of the philosophical preferences concerning the vailable interpretations of QM, we are confronted with the need to understand how the macroscopic world, where realistic values for observables, or well-defined physical properties, which might not be inherent, but are certainly observed, emerge from the quantum un-realistic world [35]. The apparent conflict between our fundamental theory of nature and the preconception of an observer-independent reality has always bothered the physical community and it seems fair to say that it remains as one of the most intriguing problems of QM [36]. As pointed out by Penrose:

> "We need a notion of physical reality, even if only a provisional or approximate one, for without it our objective universe, and thence the whole of science, simply evaporates before our contemplative gaze!" [Roger Penrose, The road to reality p. 508]

Among the historical approaches to the issue, the criticism raised by Einstein, Podolsky, and Rosen (EPR) against quantum theory [37] caused a particularly great impact. Under the premise of locality, EPR argued that incompatible observables could be simultaneously real in scenarios involving entangled states. Since QM is not able to simultaneously describe such elements of reality, it presumably is, according to EPR, an incomplete theory. On the other hand, as Bohm has shown by explicit construction [38], it is perfectly viable to have a realistic hidden-variable theory, but at the expense of local causality [36]. In recent decades, conceptual advances concerning the emergence of objective reality from the quantum substratum have been obtained by use of mechanisms such as weak measurements [39], decoherence [40], and quantum Darwinism [41, 42]. Impacting results have also been reported about the ontology of the wave function [36, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52]. More recently, with the premise that a measurement of an observable defines its reality for the measured state, Bilobran and Angelo (BA) put forward an operational scheme to assess elements of reality [53]. In a protocol involving preparation, unrevealed measurements, and quantum state tomography, they introduced a quantifier for the degree of irreality of an observable for a given state preparation. Among its many interesting properties, this measure has proven relevant in scenarios involving coherence [54], nonlocality [55], and now weak reality [36]. Also, this epistemic approach is equivalent to giving a prominent role to the notion of information, as we will discuss in this thesis.

The advent of classical and quantum information theories triggered a revolution in our understanding on the notions of probability, entropy, and information. Claude Shannon [56] showed that information is a well-defined and, most important, a measurable quantity since it can be treated like a physical property, such as mass or energy. The results developed by Shannon define practical limits on precisely how much information can be communicated between any two components of a system, as was well interpreted in the words of Landauer [57]:

> "Information is not a disembodied abstract entity; it is always tied to a physical representation. It is represented by engraving on a stone tablet, a spin, a charge, a hole in a punched card, a mark on paper, or some other equivalent. This ties the handling of information to all the possibilities and restrictions of our real physical word, its laws of physics and its storehouse of available parts."

From the idea that information is physical, we can highlight the result that there is an unavoidable entropic cost associated with the act of erasing information in an irreversibly way. This result was central for the solution proposed by Bennett to the paradox involving the Maxwell demon [58], a hypothetical creature idealized in a thought experiment made by the physicist James Clerk Maxwell. In his experiment, Maxwell devised a gas-tight container, divided into two parts by an inner wall in which there is a small door. The demon was allowed to open and close that door in a way to redistribute the particles, passing the slower and colder to one side and driving the faster and warmer to the other side of the wall, thus creating a temperature difference that would violate the laws of thermodynamic, suggesting that it would be possible to reduce the entropy of a gas without having to spend energy, only with the information of the position and the momentum of the particles, to which by assumption the so-called Maxwell demon would have access. The solution of this paradox was only possible by the physical imposition on information processing. Bennett noted that it was necessary to erase the memory of the demon who gets information about the position and momentum of the particles as to effectively close the thermodynamic cycle. The central point here was to note that there is an entropic cost to erase the memory of a system that encoded information. This example explicitly shows the interconnections among information, the laws of thermodynamics, and the definiteness of physical properties.

Despite all these efforts towards a profound understanding of the physical reality and information, too little (if any) has been achieved with regard to formal connections between elements of reality and fundamental concepts such as information and quantum correlations. The situation is not better when we try to understand the emergence of reality from the measurement process, which is a major conundrum of quantum theory [36].

Contributing to filling this gap is the central goal of this thesis. In contexts involving measurements of generic intensity, with outcomes revealed or not, we aim at deriving formal relations between BA's irreality and quantifiers of information, such as the mutual information and the von Neumann entropy, to study the role of quantum correlations for the establishment of physical reality. In particular, we want to investigate if and how entanglement, a fundamental resource for quantum technology [59, 60, 61, 62] and an important mechanism in foundational approaches [15, 36, 53, 54, 63], influences the emergence of objective reality.

However, as history has shown, entanglement is by no means the last word on quantum correlations [64]. In 2001, Ollivier and Zurek [65], and Henderson and Vedral [66] independently, discovered a type of quantum correlation that can exist even for non-entangled states. These surprising correlations are captured by a quantifier called *quantum discord* (see Ref. [67] for a review of the remarkable developments associated with quantum discord). As discussed in [64], other quantumness quantifiers also gained attention in the last decades, as for instance the EPR-steering [68, 69, 70], the geometrical quantum discord [71], the symmetric quantum discord [72] and further generalizations [73, 74], the Bell nonlocality [75, 76, 77, 78, 79, 80, 81, 82, 83], and, more recently, the realism-based nonlocality [55]. The existence of a given hierarchy underlying many of these quantifiers [55, 83] can be viewed as a theoretical evidence that the measured quantum correlations have different natures. An interesting step toward to an unifying approach for several quantum correlations measures (including quantum discord) was given by Modi *et al* in Ref. [84]. In this work, the authors show how to state a given quantum correlation quantifier as a "distance" (in terms of some entropic metric) between the state under scrutiny and a state that has been projectively measured and, therefore, does not have the corresponding quantum correlation.

The question then naturally arises as to whether one can obtain further information about quantum correlations by using weak measurements [85, 86, 87] instead of the projective ones. Since a weak measurement implies a weak disturbance on the state, the entropic distance to the undisturbed state should presumably be small. It follows from this rationale that the weak-measurement induced quantum discord should be never greater than its traditional formulation [64]. This was indeed confirmed by Li *et al* [88], who employed the Hilbert-Schmidt norm to compute a weak-measurement induced geometrical quantum discord. Surprisingly, though, by introducing weak measurements in the original procedure for the derivation of the entropic quantum discord, Singh and Pati obtained what they called a *super quantum discord* [89], a quantifier that is always greater than quantum discord. This fact was corroborated by a number of works via explicit calculations involving two-qubit states [90, 91, 92, 93]. However, as pointed out by Xiang and Jing, who also noticed the discrepancy between the super quantum discord and the weak geometrical quantum discord in contexts involving non-inertial reference frames [94], there seems to be some inconsistency in all this. The present work also aims at solving this puzzle by introducing a formulation for what we call *weak quantum discord* [64].

In particular, we want to use the tools developed in this work to learn what type of physical mechanisms can produce alterations in the degree of reality of observables and study the relationship between information and quantum correlations in those mechanisms. With the formalism developed in this work, we also want to shed some light on several foundational questions such as, the drama originally proposed by Everett concerning a quantum measurement as seen from the perspective of two distinct observers, the collapse of the wave-function, the irreversibility of measurement process and the wave-particle duality typically exhibited by quantum systems.

This thesis is structured as follows. The chapters 2 and 3 are devoted to give a pedagogical explanation of several concepts that will be used to develop our results. We start with a preliminary discussion about the conception of reality in classical physics, introduce our notation, and present several fundamental and well-established concepts of quantum theory. Next, we make a review of basic results involving classical information theory and quantum information theory. In chapter 3 we discuss several approaches for the definition of the so-called elements of reality in quantum theory, introduce the von Neumann and weak measurement models, discuss the measurement problem in the context of decoherence program and present Zurek's spin bath toy-model in order to discuss the basic features of this program. At the end of this chapter, we finally present the definition of physical reality [53] that we will use to develop our main results.

In chapter 4 we then introduce, as our first contribution, a map that conveniently interpolates between a weak and a projective measurement and a second map, defining a procedure that we call monitoring, that extends the first one to the context of unrevealed measurements. In Sec.4.2 we show that some correlating dynamics involving arbitrary-intensity interactions and some type of discard invariably lead to an increase of reality. On the other hand, we find that local irreality can be created through both revealed measurements of arbitrary intensities and unitary dynamics marked by an effective violation of some conservation law. Remarkably, we derive a complementarity relation between the information acquired by the detection system and the degree of irreality of the probed observable.

In chapter 5 we show how to consistently introduce the weak quantum discord and then prove that it is never greater than the quantum discord. In particular, it is shown that the weak quantum discord goes to zero with the intensity of the measurement. The meaning of the introduced measure is discussed in Sec. 5.1.1 and a case study is presented. As a by-product of our approach, we introduce in Sec. 5.2 the symmetrical weak quantum discord and compare the aforementioned quantifiers via the concepts of hierarchy and ordering of quantum correlations.

Finally, in chapter 6 we move to the two-observer drama proposed by Everett to discuss several aspects of the measurement problem, including the objectivity of reality, the classicality of the apparatus, the role of the reference frame, and the irreversibility of the measurement. Our approach addresses the measurement problem without invoking an external reservoir, that is, it uses only internal mechanisms of decoherence. Finally we discuss how the information-reality duality can be used to interpret the famous complementarity principle of QM as a trade-off between information and physical reality. We then close this work in 7 with our concluding remarks and future perspectives.

2 BASIC CONCEPTS AND QM THEORY

Reality is a broad conception long pursued and debated throughout human history approaching many fields of knowledge such as philosophy, physics, psychology, mathematics, art, neuroscience, and so on. Disputes on the foundational problems of QM often involve the notion of reality, without been specific to which aspect of this broad concept of reality one refers [95], so it is important to make rigorous definitions in attempt to avoid unnecessary problems. As we shall discuss in this thesis, many of the subtle aspects involving interpretative differences about the establishment of reality in the quantum realm, is related to a discussion on the role of these four "worlds", schematically illustrated below, to define reality: It is important to keep in mind, as Krizek writes in [95], that the

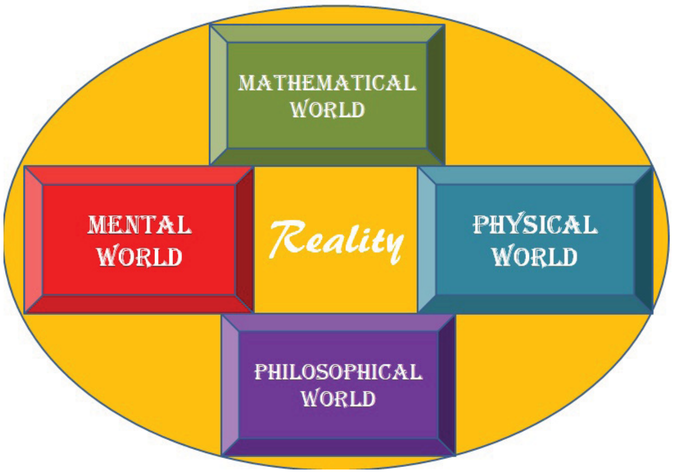

Figure 2.1: What are the fundamental ingredients that form the substratum of reality? For a more detailed discussion on the connections between these fields of knowledge and the notion of reality, we refer the reader to [6].

correspondence between reality and the formal and semantic descriptions of a physical theory can involve different concepts of this theory, as it can be related to the mathematical formalism, the assignment to measurement values, the conceptual structure, and the ontology of the theory. Since the notion of reality deeply changes in QM, first we want to provide an overview on aspects of classical realism, which are relevant to our discussion, and give an introduction to the basic ideas and concepts of the formalism of

QM. These concepts will be used in the development of our work and will be discussed more profoundly in the following chapters.

2.1 REALITY AS DEFINITENESS OF PHYSICAL PROPERTIES

There is not a unique view of physical reality, but it seems that in all of them the notion is related to the definiteness of physical quantities[1]. For example, at every instant of time we probe our surroundings through a huge number of sequential projective measurements which induce us to believe that every element of physical reality is well defined. When, for instance, we look at an object at rest on the ground, our eyes collect a bunch of photons which bring us information about the object. Because macroscopic objects are only slightly disturbed by the scattered photons, such measurements can be repeated many times yielding always the same information about the object [36]. In classical physics, a central assumption is that a measurement on a system reveals information about physical quantities without any disturbance to the system, that is, these quantities are believed to be defined prior to measurements. This notion of a rigid and unperturbed physical reality is always considered to be independent of observers. In other words, classical reality is considered as existing "since the big-bang", independently of human beings looking at it or not. This is formalized in the philosophy of science with the scientific realism, when one considers statements towards the structure of our scientific approach, and with the so-called metaphysical realism, when our assumption is prior to any statement on the connection to the structure of scientific theories [95].

Time and space are considered in Newtonian mechanics as independent aspects of objective reality. However, we know since the emergence of special relativity theory that the structure of space and time can be deeply different from the structure proposed by Newton. The rigidity of space and time and the independence between them were completely dissolved with special and general relativity theories, the later presenting us a deeper understanding on the phenomenon of gravity, seen now as a geometric property of

[1]It is important to note that we are not talking about the existence of a particular physical system, otherwise it would not make sense to talk about its physical properties.

space-time. Relativity deals with deterministic correlations between actual events, and as such, one may argue that it allows us to explain these correlations in terms of causal links between individual events that share the same light-cone in the space-time. The development of the theory of relativity was accompanied by the advent of positivistic philosophy, which strongly influenced the development of QM. Since the theory of relativity followed empiricism, which asserts that only what is measurable can be seen as real, this stance was adopted by most of the founding fathers of QM, see [95] for more details.

Today we can say that the starting point for the emergence of a radically new picture of reality in physics can be attributed to the formal description of non relativistic QM. Given the recurrent efforts made by the community towards a better understanding of the foundations of QM (see, e.g., several every-year meetings on quantum foundations), it seems fair that many physicists would regard the changes in our picture of the world given by QM as being far more revolutionary even than the curved space-time of general relativity theory [6]. It is important to note that this theory is non-relativistic in a serious way, time and space are treated in radically distinct ways, time is a one-dimensional parameter and space is a Hermitian operator.

The temporal evolution of a closed quantum state can be written following Schrödinger equation:

$$i\hbar\frac{d}{dt}\left|\Psi(t)\right\rangle = H(t)\left|\Psi(t)\right\rangle, \tag{2.1}$$

It is well known that the above evolution is linear and unitary [8, 6], that is, assuming $\left|\Psi(t)\right\rangle = U(t)\left|\Psi(0)\right\rangle$ for an initial state $\left|\Psi(0)\right\rangle$ and a unitary operator $U(t)$ such that $U^{\dagger}(t) = U^{-1}(t)$.

The process U is something familiar to physicists: a temporal evolution of a defined mathematical quantity as well as in classical mechanics. The basic difficulty of QM, and that leads to different interpretations, is a *supposed* conflict between two quantum processes, the first described by the evolution discussed above and a second that consists of a discontinuous jump (hence non-reversible) that projects the state to one possible eigenstate of the observable that is being measured. The point is that, in general, the theory is probabilistic under the observation of some physical property. Of course, classical statistics mechanics as well invokes probability theory, however, this is not in conflict with the underlying classical determinism, for, in this case, uncertainties are related to the lack of knowledge of the actual deterministic trajectories of systems which have too many inaccessible degrees of freedom. Suppose we wish to perform a measurement of an observable A written in its diagonal basis ($A = \sum_k |a_k\rangle a_k \langle a_k|$). The prescription given by QM theory to write the possible results of such a measurement in some pure state $|\psi_i\rangle$ is given by the projector operator $P_k = |a_k\rangle\langle a_k|$ which has the following properties

$$\sum_k P_k = \mathbb{1}, \tag{2.2}$$

$$P_k^2 = P_k, \tag{2.3}$$

and

$$P_j P_k = 0. \tag{2.4}$$

According to the postulate of reduction, the state of the system immediately after obtaining the result k is

$$|\psi_i^k\rangle = \frac{P_{A_k}|\psi_i\rangle}{\sqrt{\langle\psi_i|P_{A_k}^\dagger P_{A_k}|\psi_i\rangle}}. \tag{2.5}$$

The conditional probability $p(k|i)$ of obtaining the eigenvalue a_k upon a measurement of A given the initial state $|\psi_i\rangle$ is

$$p(k|i) = \langle\psi_i|P_{A_k}^\dagger P_{A_k}|\psi_i\rangle = Tr(P_{A_k}^\dagger P_{A_k}|\psi_i\rangle\langle\psi_i|). \tag{2.6}$$

If we consider now that we don't have knowledge about what $|\psi_i\rangle$ we have from an ensemble of possible pure states, such that we attribute a probability $p(i)$ for each possible state, we can use the law of total probability to write,

$$p(k) = \sum_i p(k|i)p(i) = \sum_i p(i)Tr(P_{A_k}^\dagger P_{A_k}|\psi_i\rangle\langle\psi_i|) = Tr(P_{A_k}^\dagger P_{A_k}\sum_i p(i)|\psi_i\rangle\langle\psi_i|). \tag{2.7}$$

The term $\rho = \sum_i p(i)|\psi_i\rangle\langle\psi_i|$ is also a possible quantum state as we shall discuss. This representation is very important to describe situations with insufficient information about the state and to this thesis since we will fix our notion of physical reality in a quantifier that deals with density operators and their observables.

2.2 QUANTUM STATE

The discussion about the meaning of quantum states is often presented as a conflict between two fundamentally distinct approaches. The first speaks of "states of reality" that are independent of any possible empirical access, and, implicitly, presupposes the existence of such states. Such an approach is known in the literature as the ontological interpretation of quantum states. The second, known as epistemic interpretation, refers to observations, the amount of data that a certain observer has to know to deduce the possible results and their respective probability of occurrence in a particular type of experiment. Regardless of the interpretation given to a quantum state, the normalized vectors $|\psi\rangle$ are no longer appropriate if we are dealing with a statistical mixture of systems, being necessary the introduction of the concept of density operators. The pure state in that representation is written as $\rho = |\psi\rangle\langle\psi|$.

In general, the density operator is a Hermitian, positive semidefinite matrix that has also a unity trace, i.e:

(i) Hermiticity:

$$\rho^\dagger = \rho. \tag{2.8}$$

(ii) Unity of trace:

$$Tr[\rho] = 1. \tag{2.9}$$

(iii) Positivity: Given an arbitrary vector, $|\xi\rangle \in \mathcal{H}^{(n)}$ we have:

$$\langle\rho\rangle_\xi = \langle\xi|\,\rho\,|\xi\rangle \geq 0, \tag{2.10}$$

where $\langle\xi|$ is a vector that belongs to the dual space of $\mathcal{H}^{(n)}$.

A subtle and pertinent question, would be about the nature of the statistical uncertainty associated with a quantum state. Would this uncertainty be of a subjective nature (ignorance of the observer about an already-established value of a physical properties) or an irreducibly objective character?

The density matrix, or density operator, in the originally form proposed by Von Neumann and Landau is known in the literature as *proper* density matrix [96]. It is formulated through an analysis known as quantum ensemble, where classical probabilities (in the sense that the ignorance in question is subjective) are used together with quantum probabilities (related to a fundamental indefiniteness) through the construction of the density matrix. However, there is another distinct way of presenting the density matrix, known in literature as *improper* density matrix. This takes on a purely quantum approach to deal with this definition, without invoking a classical statistical factor. In the following, we describe both approaches succinctly. As we shall see, these two formalizations imply the same mathematical object. At the end of this work, we intend to analyze this equivalence from an informational point of view.

2.2.1 Proper density matrix

Suppose an observer has access to a very large number of N quantum systems of the same type, and each of these systems belongs to a space $\mathcal{H}^{(m)}$ that we will suppose to be m-dimensional. Considering the observable $\{|o_j\rangle\}$ with $(j = 1, 2, ..., m)$, we have that $O = \sum_j |o_j\rangle\, o_j \langle o_j|$. Assuming that, at a given moment the N quantum systems are partitioned into P sub partitions with n_α systems in a given state $|\psi_\alpha\rangle$ with a relative fraction $\lambda_\alpha = \frac{n_\alpha}{N}$, where α it varies from one to the total number of partitions. It can be clearly seen that:

$$\sum_{\alpha=1}^{P} \lambda_\alpha = 1 \qquad \text{with } 0 < \lambda_\alpha < 1. \tag{2.11}$$

Repeating the same experiment many times, that is, in an ensemble where $N \to \infty$, one can calculate the average of the expectation value of the observable by the following average:

$$\frac{1}{N} \sum_{\alpha=1}^{P} n_\alpha \langle \psi_\alpha | O | \psi_\alpha \rangle = \sum_{\alpha=1}^{P} \lambda_\alpha \langle O \rangle_{\psi_\alpha}, \tag{2.12}$$

where each “purely quantum”expectation value $\langle O \rangle_{\psi_\alpha}$ is weighted on average by a “classical"statistical factor λ_α. Note that by the linearity of the trace, we can write

$$\sum_{\alpha=1}^{P} \lambda_\alpha \langle\psi_\alpha| O |\psi_\alpha\rangle = \sum_{\alpha=1}^{P} \lambda_\alpha Tr[|\psi_\alpha\rangle \langle\psi_\alpha| O] = Tr[\rho O], \tag{2.13}$$

such that

$$\rho = \sum_{\alpha=1}^{P} \lambda_\alpha |\psi_\alpha\rangle \langle\psi_\alpha| \tag{2.14}$$

is the *density operator* of the ensemble. In this way, we can define the expectation value of ensemble in terms of:

$$\langle O\rangle_\rho = Tr[\rho O]. \tag{2.15}$$

Note that the density operator obeys the previously described properties:

(i) Trace unitarity

$$Tr[\rho] = Tr\left[\sum_{\alpha=1}^{N} \lambda_\alpha |\psi_\alpha\rangle \langle\psi_\alpha|\right] = \sum_{\alpha=1}^{N} \lambda_\alpha \langle\psi_\alpha|\psi_\alpha\rangle = \sum_{\alpha=1}^{N} \lambda_\alpha = 1. \tag{2.16}$$

(ii) Hermiticity:

$$\hat{\rho}^\dagger = \hat{\rho}. \tag{2.17}$$

(iii) Semi-positivity: For an arbitrary state, $|\xi\rangle \in H^{(m)}$ we have:

$$\langle\hat{\rho}\rangle_\xi = \langle\xi| \hat{\rho} |\xi\rangle = \sum_{\alpha=1}^{N} \lambda_\alpha |\langle\psi_\alpha|\xi\rangle|^2 \geq 0, \text{ because } 0 \leq \lambda_\alpha \leq 1. \tag{2.18}$$

It is important to note that in this formulation, the classical probability is not objective, but *subjective*, because it appears due to an ignorance about which sub-ensemble the system belongs. Another more realistic way of building the same density matrix above would be that an agent (Alice) constructs a mechanism capable of producing a state $|\psi_\alpha\rangle$ with a known probability λ_α to produce this state. After this, Alice delivers the system to a second party (Bob) along with the detailed information of the distribution λ_α. Without information about the specific state that was produced, the best descriptor for Bob is represented by ρ.

2.2.2 Improper density matrix

In the second approach we will assume a space $\mathcal{H}^{(u)} = \mathcal{H}_A^{(m)} \otimes \mathcal{H}_B^{(n)}$ with dimension $u = m \times n$, given by the tensor product of a space of dimension m (the space that represents a system that we have access to) and a space of dimension n (which represents a system which, for some reason, we do not have physical access to). Consider $|\Psi\rangle \in \mathcal{H}^{(u)}$ a pure state of the total space and an observable O accessible only to the subsystem $\mathcal{H}_A^{(m)}$, then it is reasonable to define the expectation value restricted to the subsystem $\mathcal{H}_A^{(m)}$ as $\langle\Psi| O \otimes \mathbb{1} |\Psi\rangle$. We can then associate with each pure state $|\Psi\rangle$ of the *total system*, a density operator $\rho_{|\Psi\rangle}$ of the subsystem $\mathcal{H}_A^{(m)}$ through the relation

$$Tr[\rho_{|\Psi\rangle}O] = \langle\Psi| O \otimes \mathbb{1} |\Psi\rangle \qquad . \tag{2.19}$$

As in the previous case, it is possible to show that $\rho_{|\Psi\rangle}$ also obeys the three properties that define the density operator. Also, note that the above procedure suggests the following application, known as partial trace, which associates each operator acting on $\mathcal{H}^{(u)}$ with another that acts only on $\mathcal{H}^{(m)}$:

$$\begin{aligned} \mathcal{H}^{(u)} \otimes \bar{\mathcal{H}}^{(u)} &\to \mathcal{H}_A^{(m)} \otimes \mathcal{H}_A^{(m)} \\ \sum_i \lambda_i A_i \otimes B_i &\to \sum_i \lambda_i (TrA_i) \otimes B_i, \end{aligned} \tag{2.20}$$

such that A_i and B_i are basis of $\mathcal{H}_A^{(m)} \otimes \bar{\mathcal{H}}_A^{(m)}$ and $\mathcal{H}_B^{(n)} \otimes \bar{\mathcal{H}}_B^{(n)}$ respectively. The partial trace is then defined as

$$\sum_i \lambda_i (TrA_i) \otimes B_i \equiv Tr_A(\sum_i \lambda_i A_i \otimes B_i). \tag{2.21}$$

In addition to properly describing states of systems that is not well known, this formalism is also a good tool for describing subsystems constituents of composite systems.

Such description is provided by the reduced density operator defined as:

$$\begin{aligned} \rho_B &\equiv Tr_A[\rho_{AB}], \\ \rho_A &\equiv Tr_B[\rho_{AB}]. \end{aligned} \tag{2.22}$$

It is interesting to note that, in this case, the improper density matrix is obtained in a different way. The "ignorance" on the state of the system seems to have a completely different notion, where information lost due to the partial trace is "hidden" in the correlations of the global state. However, the mathematical object is exactly the same as in the case of proper density matrix. For more detailed discussion on that we refer the reader to [96]. In the following, we discuss the quantum axioms in the context of density operators.

2.3 TEMPORAL EVOLUTION OF DENSITY OPERATORS

Regarding the temporal evolution of the density operator, it can be seen that the same evolves in time according to a specific equation. As the temporal evolution of closed quantum state is unitary, we can write:

$$|\Psi(t)\rangle = U(t)\,|\Psi(0)\rangle\,, \tag{2.23}$$

and assuming that each quantum state $|\psi_i\rangle$ evolves in time with $U(t)$, we can write the result for the ensemble in the following way,

$$\rho(0) = \sum_i p_i|\psi_i\rangle\langle\psi_i| \overset{U(t)}{\to} \sum_i p_i U(t)|\psi_i\rangle\langle\psi_i|U^\dagger(t) = U(t)\rho U^\dagger(t). \tag{2.24}$$

So

$$\frac{d\rho(t)}{dt} = \frac{dU(t)}{dt}\rho(0)U^\dagger(t) + U(t)\rho(0)\frac{dU^\dagger(t)}{dt}, \tag{2.25}$$

we can write a differential equation for $U(t)$

$$i\hbar\frac{dU(t)}{dt} = H(t)U(t), \tag{2.26}$$

and then

$$\frac{d\rho(t)}{dt} = \frac{1}{i\hbar}H(t)U(t)\rho(0)U^{\dagger}(t) - \frac{1}{i\hbar}U(t)\rho(0)U^{\dagger}(t)H(t), \tag{2.27}$$

such that H is the Hamiltonian of the system. Thus the time evolution of a density matrix is given by the Liouville-von Neumann equation :

$$i\hbar\frac{d\rho(t)}{dt} = [H(t), \rho(t)], \tag{2.28}$$

Summarizing, isolated systems evolve in time via a unitary operator $U(t)$ with $U^{\dagger}(t) = U^{-1}(t)$, according to:

$$\rho(t) = U(t, t_0)\rho(t_0)U^{\dagger}(t, t_0). \tag{2.29}$$

2.4 MEASUREMENTS ON DENSITY OPERATORS

2.4.1 Projective measurements

Suppose we wish to perform a measurement of an observable A written in its diagonal basis ($A = \sum |a_k\rangle a_k \langle a_k|$). The prescription given by QM theory to write the possible results of such a measurement in some pure state $|\psi_i\rangle$ is given by the projector operator $A_k = |a_k\rangle\langle a_k|$. We know from the postulate of reduction that a measurement of A with a result a_k results in

$$|\psi_i^k\rangle = \frac{A_k|\psi_i\rangle}{\sqrt{\langle\psi_i|A_k^{\dagger}A_k|\psi_i\rangle}} \tag{2.30}$$

By noting that,

$$p(k) = \sum_i p(k|i)p(i) = \sum_i p(i)Tr(A_k^{\dagger}A_k|\psi_i\rangle\langle\psi_i|) = Tr(A_k^{\dagger}A_k\sum_i p(i)|\psi_i\rangle\langle\psi_i|), \tag{2.31}$$

and from Baye's rule we can write that after a measurement of this type in an ensemble of pure states $|\psi_i\rangle$ we will have the following density matrix ρ_k

$$\rho_k = \sum_i p(i)\frac{A_k|\psi_i\rangle\langle\psi_i|A_k^{\dagger}}{Tr(A_k^{\dagger}A_k\sum_i p(i)|\psi_i\rangle\langle\psi_i|)} = \frac{A_k\rho A_k^{\dagger}}{Tr(A_k^{\dagger}A_k\rho)}. \tag{2.32}$$

This is the projection postulate written to density matrices. Now, we expand this result to a $N-$partite state. Consider ρ in $\mathcal{H}_A \otimes \mathcal{H}_B$ a $N-$partite state where $\mathcal{H}_B = \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_N$:

(i) Strong measurements with known results.

If an observable $A_1 = \sum_k a_{1k} A_{1k}$ on $\mathcal{H}_A$ is measured, with an output a_{1k}, for a system initially prepared in $\mathcal{H}_A$, the transition to the *colapsed* state is given by:

$$\mathcal{C}_{k|A_1}(\rho) = \frac{A_{1k}\rho A_{1k}}{p_k} = A_{1k} \otimes \rho_{\mathcal{B}|k} \ , \tag{2.33}$$

where $\rho_{\mathcal{B}|k} = \langle a_{1k}|\rho|a_{1k}\rangle / p_k$ and $p_k = \mathrm{Tr}[(A_{1k} \otimes \mathbb{1}_{\mathcal{B}})\rho]$. Note that

$$\mathcal{C}_{k|A_1}(\mathcal{C}_{k|A_1}(\rho)) = \mathcal{C}_{k|A_1}(\rho), \tag{2.34}$$

which means that two revealed projective measurements from the same observable exhibit the repeatability property of the result. Also, note that for two orthogonal projections,

$$\mathcal{C}_{j|A_1}(\mathcal{C}_{k|A_1}(\rho)) = 0. \tag{2.35}$$

(ii) Strong measurements with unknown results.

When A_1 is measured, for a system with preparation ρ, but the result is not revealed, the final state of the system is predicted to be $\mathcal{C}_{k|A_1}(\rho)$ with probability $p_k = Tr(A_{1k}\rho A_{1k})$. In this case, the post-measurement state is:

$$\sum_k p_k \mathcal{C}_{k|A_1}(\rho) = \sum_k A_{1k}\rho A_{1k} = \Phi_{A_1}(\rho), \tag{2.36}$$

The map above is called a completely positive and trace preserving map (CPTP), since it maps Hermitian operators to Hermitian operators, positive operators to positive operators and preserves the trace. According to Stinespring's dilation theorem, any CPTP map can be built from the basic operations of tensoring with a second system in a pure state, a global unitary transformation, and partial trace over the second system. Any quantum operation can be thought of as arising from a unitary evolution on a larger (dilated) system $\rho \in \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{X}}$. The auxiliary system $\mathcal{H}_{\mathcal{X}}$ is usually called the ancillary system[97]

$$\rho_{\mathcal{S}}(t) = \mathrm{Tr}_{\mathcal{X}}\left[U(t)\,\rho_{\mathcal{S}} \otimes |x_0\rangle\langle x_0|\,U^{\dagger}(t)\right]. \tag{2.37}$$

More explicit,

$$\rho_{\mathcal{S}}(t) = \sum_k \langle x_k|U(t)\rho_{\mathcal{S}} \otimes |x_0\rangle\langle x_0|\,U^{\dagger}(t)|x_k\rangle = \sum_k M_k(t)\rho_{\mathcal{S}}M_k^{\dagger}(t), \tag{2.38}$$

where $M_k(t) = \langle x_k|U(t)|x_0\rangle$ is an operator that acts on $\mathcal{H}_{\mathcal{S}}$. This equation is known as operator-sum representation and it is useful for characterizing the dynamics of the system of interest without considering the total space. Stinespring's representation comes with a bound on the dimension of the ancillary system, and is unique up to unitary equivalence.

2.4.2 POVM's

The more general form of representing a measurement in QM is through the set of operators $\{M_k\}$ (*Positive Operator-Valued Measure*) which is prescript with the following properties:

(i) $M_k^{\dagger}M_k \geq 0$.

(ii) $\sum_k M_k^{\dagger}M_k = \mathbb{1}$.

The Naimark extension (or dilation) theorem says that any POVM measurement can be realized as a projective measurement on the ancillary system $\mathcal{H}_{\mathcal{X}}$, where realized means that the probabilities of the measurement outcomes are the same. This is, given a POVM that acts on $\mathcal{H}_{\mathcal{S}}$, it is possible to find an auxiliary system that acts on $\mathcal{H}_{\mathcal{X}}$ such that there is a unitary U and a projective operator $\{P_k\}$ for which the following condition holds

$$Tr[M_k^{\dagger}M_k\rho] = Tr[P_kU(\rho \otimes |0\rangle\langle 0|)U^{\dagger}]. \tag{2.39}$$

With that we can write the normalized state of a quantum system after this operation

$$\rho_f = \frac{M_k\rho M_k^{\dagger}}{p_k}, \tag{2.40}$$

with $p_k = Tr(M_k \rho M_k^\dagger)$ being the probability of getting a result k. This result tells us that a POVM can always be interpreted as being the result of a projective measurement in an auxiliary system such that it has interacted with the system of interest at a time prior to the measurement [97].

2.5 CLASSICAL AND QUANTUM CORRELATIONS

2.5.1 Entropy of a random variable

Claude Shannon established in 1948 two central results for the classical theory of information [56]. The first result tells us how much one message can be compressed, that is, how much redundancy exists in a finite sequence of bits. Here, we are considering a message as a finite sequence of n symbols, chosen from an ensemble X with k different symbols:

$$X = \{x_0, x_1, ..., x_{k-1}\} \tag{2.41}$$

The second result tell us at what rate we can communicate in a channel with noise without loss of information during the process, that is, how much redundancy can be embedded in a message so that it is protected from possible errors. Both questions relate to how unexpected is the next letter of a particular message on average. If we consider a message of n letters statistically independent of each other, we can say that a specific letter x will typically appears around np_x times. The profound result of Shannon was to verify that the expression

$$H(X) = -\sum_x p_x \ln p_x \tag{2.42}$$

provides a way to quantify how much redundancy is associated with a particular message by telling us how much uncertainty we have with a specific letter that forms the message, being recognized as Shannon's entropy for the ensemble $X = (x, p_x)$.

Suppose now X and Y two random variables. How the information contained in X is related to Y? To answer this question it is necessary to introduce two concepts, conditional entropy and mutual information. The first is related to the conditional probability given by the Bayes rule and tells us what is the uncertainty about some variable, given that we have information about the other. The second tells us how much information is shared by both systems. The joint probability distribution of getting outcomes x and y, respectively, is $p_{x,y}$. The Shannon entropy

$$H(X,Y) = -\sum_{x,y} p_{x,y} \ln p_{x,y} \tag{2.43}$$

quantifies the joint ignorance that an observer has about these random variables. On the other hand,

$$H(X) = -\sum_{x} p_{x} \ln p_{x} \tag{2.44}$$

quantify the amount of ignorance specifically associated with each variable, where $p_x = \sum_y p_{x,y}$ denotes the marginal probability distribution associated with the variable X. The classical notion of mutual information, which is formally written as

$$I_{X:Y} = H(X) + H(Y) - H(X,Y), \tag{2.45}$$

encapsulates the amount of information about Y that is codified in X, and vice-versa. In this capacity, mutual information is a measure of correlations. Interestingly, there is another formula for the mutual information which makes explicit reference to the measurement process:

$$J_{X:Y} = H(X) - \sum_{y} p_y H(X|y), \tag{2.46}$$

where

$$H(X|y) = -\sum_{x} p_{x|y} \ln p_{x|y} \tag{2.47}$$

is the entropy of X conditioned to the outcome y and

$$H(X|Y) = \sum_{y} p_y H(X|y) \tag{2.48}$$

is the (average) conditional entropy. Now, using the very definition of conditional probability, $p_{x|y} = p_{x,y}/p_y$, we have

$$H(X|Y) = -\sum_y p_y \left(\sum_x \frac{p_{x,y}}{p_y} \ln \frac{p_{x,y}}{p_y} \right) = -\sum_{x,y} p_{x,y} \ln p_{x,y} + \sum_y p_y \ln p_y, \tag{2.49}$$

so it is clear that,

$$J_{X:Y} = H(X) - H(X|Y) = H(X) + H(Y) - H(X,Y) = I_{X:Y}. \tag{2.50}$$

Therefore, we have two equivalent expressions that define the *total correlations* between two random variables. We can schematically represent the procedure adopted above through a diagram:

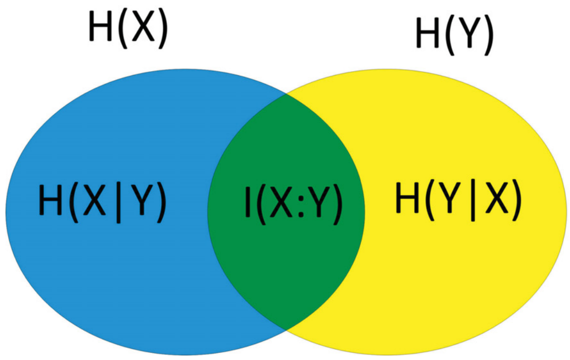

Figure 2.2: Illustration, via Venn's diagram for the links between Shannon's entropy and mutual information

For transmission and processing of quantum information, we must take into account the quantum properties of the systems, which give rise to correlations that differ from the correlations we have described above. In the following we generalize this idea to the quantum context.

2.5.2 Entropy of a quantum state

Entropy is a central concept in quantum information theory, as it is related with how much uncertainty there is associated with a state of a quantum physical system. A minimum requirement that must be made for such a function is that it is additive for independent events. One of the most commonly used entropic functions to define the entropy of quantum states is the von Neumann entropy, which is defined for a state ρ as:

$$S(\rho) \equiv -Tr(\rho \ln \rho). \tag{2.51}$$

Since ρ is Hermitian it can be diagonalized. In its eigenbasis $\{|u_j\rangle\}$ it reads

$$\rho = \sum_j p_j |u_j\rangle \langle u_j|, \tag{2.52}$$

that is

$$S(\rho) = -\sum_j p_j \ln p_j = H(U), \tag{2.53}$$

where $H(U)$ is the Shannon entropy for the ensemble $U = \{u_j, p_j\}$. The entropies of Shannon and von Neumann will be used throughout this work and for more details on the construction of such quantities we refer the reader to [59]. Below we list some properties of the von Neumann entropy which will prove to be relevant to this work

(i) The entropy is non-negative, $S(\rho) \geq 0$, being null only for pure states.

(ii) It is upper bounded by $\ln d$, where d the dimension of space Hilbert associated with the state. In this case, the entropy is equal to $\ln d$ only if the system is in the maximally mixed state $\frac{\mathbb{1}}{d}$, where all states are equally likely.

(iii) Unitary invariance, $S(U\rho U^\dagger) = S(\rho)$.

(iv) In a bipartite quantum system described by ρ_{AB}, it holds that $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$, and $S(\rho_{AB}) \geq |S(\rho_A) - S(\rho_B)|$ (Araki-Lieb inequality). In the first relation, the equality occurs if, and only if, the state is uncorrelated, that is, $\rho_{AB} = \rho_A \otimes \rho_B$.

(v) For $\sum_i p_i = 1$ and density operator ρ_i, it holds that $S(\sum_i p_i \rho_i) \geq \sum_i p_i S(\rho_i)$ (concavity).

(vi) For tripartite quantum states, it holds that $S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC})$ (strong subadditivity).

In what follows, we discuss how von Neumann entropy can be used to quantify entanglement for bipartite pure states.

2.5.3 Entanglement for pure states

Entanglement is modernly defined as a class of correlations that cannot be created via local operations and classical communication(LOCC) [98]. Local operations are operations made in only one part of a global system, that is, they are performed locally in one of the subsystems of the composite system, while classical communication refers to the transmission of information using classical devices. The states originated from LOCC are separable since there are no quantum mechanical interactions between the systems, that is, it is possible to factorize the compound state as a product of its constituent states. For pure bipartite states, we say that a state Ψ is separable if it can be written in the form $|\Psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$, otherwise the state of the system is said entangled. In this way, entanglement is defined as the non-separability of a quantum state. If a state can not be generated through LOCC it is non-separable and hence entangled. The use of von Neumann entropy to quantify entanglement was introduced by Bennett et al [99]. This amount was defined as entanglement entropy and it is calculated for a bipartite pure state ρ_{AB} as

$$E(\rho_{AB}) = S(\rho_{A(B)}), \tag{2.54}$$

where $\rho_{A(B)} = Tr_{B(A)}[\rho_{AB}]$. This measure satisfies a number of conditions which will be listed below:

(i) If ρ_{AB} is separable then $E(\rho_{AB}) = 0$.

(ii) The entanglement of a maximally entangled pure state ρ_{AB} with dimension d^2 is given by $E(\rho_{AB}) = \ln d$.

(iii) $E(\rho_{AB})$ can not increase under LOCC operations.

(iv) In the limit as the distance between two states tends to zero, the difference between their entanglements has to go to zero, that is, if $||\rho - \sigma|| \to 0$ with $||\xi|| = \sqrt{Tr[\xi^\dagger \xi]}$ the Hilbert-Schmidt norm, we have $E(\rho) - E(\sigma) \to 0$.

(v) This measure respects also additivity, subadditivity, and convexity.

For more details, we refer the reader to [99]. There are several works in the literature that are dedicated to studying the quantum correlations for pure bipartite states. The extension of a measure of entanglement to the case of mixed states is not simple and does not always give us a sufficient and necessary condition to detect entanglement. When we generalize the concept of quantum correlation for mixed states is the quantum discord that appears as a more fundamental type of correlation, since it is possible to identify states that do not possess entanglement but have discord.

2.5.4 Quantum Discord

Quantum discord (QD) originally appeared as the breakdown, at the quantum level, of a given equivalence in the classical information theory between two equivalent forms of the mutual information, namely

$$J_{X:Y} = H(X) - H(X|Y) = H(X) + H(Y) - H(X,Y) = I_{X:Y}. \tag{2.55}$$

In 2001, Ollivier and Zurek [65] noted that such equivalence cannot be established in the quantum domain. On the one hand, the quantum counterpart of the mutual information (2.45) can be directly written as

$$I(\rho) = S(\rho_{\mathcal{A}}) + S(\rho_{\mathcal{B}}) - S(\rho), \tag{2.56}$$

where $S(\rho) = -\mathrm{Tr}(\rho \ln \rho)$ is the von Neumann entropy, ρ is a density operator acting on the composite space $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$, and $\rho_{\mathcal{A}(\mathcal{B})} = \mathrm{Tr}_{\mathcal{B}(\mathcal{A})}\rho$ is the reduced state acting on the subspace $\mathcal{H}_{\mathcal{A}(\mathcal{B})}$. On the other hand, to devise the counterpart of the formula (2.46) one needs to specify measurement operators and then the pertinent conditional entropy. Ollivier and Zurek proposed to use the set $\{B_b\}$ of projectors of an observable $B = \sum_b bB_b$ acting on $\mathcal{H}_{\mathcal{B}}$. The second form of the mutual information was then proposed to be

$$J(\rho) = S(\rho_{\mathcal{A}}) - \sum_b p_b S(\rho_{\mathcal{A}|b}), \tag{2.57}$$

where $\rho_{\mathcal{A}|b} = \mathrm{Tr}_{\mathcal{B}}[(\mathbb{1} \otimes B_b)\rho(\mathbb{1} \otimes B_b)]/p_b$, and $p_b = \mathrm{Tr}[(\mathbb{1} \otimes B_b)\rho(\mathbb{1} \otimes B_b)]$. The second term on the right-hand side of Eq. (2.57) is the quantum counterpart of the condition entropy $H(X|Y)$. Now the crux comes. The forms (2.56) and (2.57) are not equivalent and the minimum deviation $I(\rho) - J(\rho)$ defines the so-called QD:

$$D_{\mathcal{B}}(\rho) := \min_{B} \Big[\sum_{b} p_b S(\rho_{\mathcal{A}|b}) + S(\rho_{\mathcal{B}}) - S(\rho) \Big]. \tag{2.58}$$

It is clear that the QD is, by construction, a measure of quantum correlations. By the same year, introducing the notion of classically accessible correlations, $\mathcal{C}(\rho) = \max_B J(\rho)$, Henderson and Vedral [66] observed that one can write Eq. (2.58) as

$$D_{\mathcal{B}}(\rho) = I(\rho) - \mathcal{C}(\rho), \tag{2.59}$$

a form that allows us to interpret mutual information as the sum of purely quantum and purely classical correlations.

Later on, Rulli and Sarandy [72] gave to QD an alternative shape. Taking the completely positive trace-preserving map

$$\Phi_B(\rho) := \sum_b (\mathbb{1} \otimes B_b)\rho(\mathbb{1} \otimes B_b) = \sum_b p_b \rho_{\mathcal{A}|b} \otimes B_b \tag{2.60}$$

and the identity $S(\Phi_B(\rho)) = S(\Phi_B(\rho_{\mathcal{B}})) + \sum_b p_b S(\rho_{\mathcal{A}|b})$ (see the joint-entropy theorem [59]), those authors wrote the QD as

$$D_{\mathcal{B}}(\rho) = \min_B \Big[I(\rho) - I(\Phi_B(\rho)) \Big]. \tag{2.61}$$

Besides allowing for the generalization of the notion of QD to multipartite states in a symmetrical way, which was the main goal of Rulli and Sarandy, this form admits an interesting interpretation for QD. To see this we first note that $\Phi_B(\rho)$ can be viewed as a state that has undergone an *unrevealed projective measurement* of the observable B (see [53, 36] for more details). It follows that QD is the minimum "distance" between ρ and the projectively disturbed non-discordant state $\Phi_B(\rho)$, where the "metric" used is the mutual information. This is in conceptual agreement with the unified view discussed in Ref. [84]. Of course, other metrics can be (and have been) used, including geometric ones [64, 71, 74].

3 REALITY AND THE MEASUREMENT PROBLEM

3.1 ELEMENTS OF REALITY IN QUANTUM THEORY

In their celebrated work of 1935 "Can Quantum-Mechanical Description of Physical Reality be Considered Complete", Einstein, Podolsky, and Rosen (EPR) were the pioneers in their attempt to define the notion of "elements reality" in a physical theory. It is worth mentioning that EPR is often cited to evoke the authority of Einstein, but he did not liked the published text at all, so it is important to keep in mind that what we shall be discussing is indeed Podolsky's final version of the paper, as Einstein once wrote [100]:

> "For reasons of language this paper was written by Podolsky after several discussions. Still, it did not come out as well as I had originally wanted; rather, the essential thing was, so to speak, smothered by formalism."

Nevertheless, as discussed by Fine, EPR's article is among the top ten of all papers ever published in Physical Review journals and due to its fundamental role in the development of Bell's work and to the emergence of quantum information theory, it is also near the top of currently "hot" papers [100].

For EPR, it is necessary to distinguish between the objective reality of physical concepts under what a physical theory operates. In this sense, it is worth noting that although the name "elements of reality"seems to have an ontological significance, in the present work and historically [39], the definitions of elements of physical reality are epistemological concepts. As pointed by Fine, EPR wondered whether it was possible, at least in principle, to ascribe certain properties to a quantum system in the absence of measurement. For example, would the decay of an atom occurs at a definite moment in time even though such a definite decay time is not implied by the quantum state function? That is, EPR had the intention to ask whether the formalism of QM provides a description of quantum systems that is complete and, with that, began to probe how robust the quantum theory was tied to irrealism and indeterminism [100].

From some basic assumptions, EPR reach the conclusion that the description given by the QM theory is incomplete. To arrive at this conclusion, EPR proposed a *necessary* condition to consider a physical theory complete:

Completeness-EPR: "Every element of physical reality must have a correspondence in theory."

Then, they define a *sufficient* condition for the notion of elements of reality:

Elements of Reality-EPR: "If, without disturbing the system in any way, we can predict with certainty (that is, with a probability equal to unity) the value of a physical quantity, then there is an element of physical reality corresponding to this quantity".

For uncorrelated systems, this criterion makes clear reference to eigenstates. It is also possible to note that they have assumed a strong version of relativistic causality [39], that is, no space-like event could influence the results of the experiments. The EPR elements of reality are related to the results of measurements that we can determine by using other measurements (performed in a separate region characterized as a space-like event). In the original argument, the physical properties analyzed are position and momentum, but an easier version was later developed by Bohm in terms of spin components. The EPR argument on elements of reality, when applied to spins, can be stated as follows: Suppose we have a pair of entangled states in a singlet state

$$|\phi\rangle_{\mathcal{AB}} = \frac{1}{\sqrt{2}}(|0\rangle_{\mathcal{A}}|1\rangle_{\mathcal{B}} - |1\rangle_{\mathcal{A}}|0\rangle_{\mathcal{B}}). \tag{3.1}$$

After the formation of the entangled pair, both systems propagate in opposite directions, so that we can classify them according to their future spatial position A and B. Suppose then that after the process described above, the spin in the spatial region A is measured in some direction. After the detection of the particle located in the spatial region A, we can conclude immediately what is the spin in the same direction for the state of the particle that is in the spatial region B, which is always perpendicular to the spin of the particle A. Note that this conclusion is independent of the spatial distance between particle A and B after their interaction. Upon the measurement of one of the systems, the spin of the other can be anticipated with certainty, then according to EPR, there must be an element of physical reality corresponding to this physical quantity. A deep consequence of that argument is that the exact prediction of a measurement result on a distant system, forcefully invites us to interpret this result as the mere manifestation of a pre-existing value of a physical property of the object since both systems did not interact while the measurement was carried out in the other separated region. If we combine this with the fact that we can measure the spin along any direction and the correlation is the same, this implies that also incompatible observables have their elements of reality existing independent by the measurement that is carried out in the other location. This is the crucial point of the argument because since QM generally does not give us such a value (for observables that do not commute), EPR concluded that it is an incomplete description of reality.

In his reply to EPR, Bohr [101] argues in terms of his complementarity principle, according to which the elements of reality of incompatible observables cannot be established in the same experiment, but only through mutually excluding experimental arrangements. Thus, one cannot claim simultaneous reality for incompatible observables within the same experimental instance, even when entangled states are involved. In addition, for Bohr one cannot speak of the nature of microscopic systems before making a measurement. This perspective refutes EPR's rationale and elects the correlations generated in the experimental setup as the mechanism responsible for the establishment of physical reality (see Refs. [53, 54] for related discussions). What the Einstein-Bohr debate initialized was an deep debate on the very nature of reality. In the same year, Ruark pointed out that EPR's conclusion derived from the adoption of a criterion that "is directly opposed to the view held by many theoreticians, that a physical property of a given system has reality only when it is actually measured" [102].

Inspired by EPR's criterion, Redhead proposes [103]:

Elements of Reality-Redhead: "If we can predict with certainty, or at any rate with probability one, the result of measuring a physical quantity at time t, then, at time t, there exists an element of reality corresponding to this physical quantity and having value equal to the predicted measurement result."

Although apparently similar to EPR's definition, this one is intended to soften the condition on the relativistic causality hypothesis [39].

Realizing that a point common to all of these definitions is the relation with actual results of quantum measurements, Vaidman then proposes that "for any definite result of a measurement there is a corresponding element of reality" [39]. Regarding "definitive result" as the definite shift of the probability distribution of the pointer variable, he suggests the following definition of elements of reality:

Elements of Reality-Vaidman: "If we are certain that a procedure for measuring a certain variable will lead to a definite shift of the unchanged probability distribution of the pointer, then there is an element of reality: the variable equal to this shift."

In this perspective, the expectation value of an observable and the weak value of an observable for a given pre- and post-preparation begin to represent elements of physical reality as well, thus generalizing EPR definition, as we discuss in detail in the next two sections. Vaidman made this definition to include the results of weak measurements as also producing an element of reality in QM.

To present the weak measurement concept we need first to discuss the so-called von Neumann's ideal measurement model.

3.2 MEASUREMENT MODEL

Let $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_M$ be the joint Hilbert space associated with the physical system S to be measured and the measurement system M. Consider also that we want to measure a discrete variable defined by the observable $O = \sum_i |o_i\rangle o_i \langle o_i|$ and the apparatus, will only be described by the movement of the center of mass of a pointer, where we disregard any internal variables of the system. Thus, we can choose, as basis for the subsystem $\mathcal{H}_M$ the usual eigenstates of position and momentum $\{|x\rangle\}$, $\{|p\rangle\}$ in such a way that the relations of completeness and the internal product of the elements of a basis with the elements of the other is expressed as:

$$\int_{-\infty}^{+\infty} |x\rangle\langle x|dx = \int_{-\infty}^{+\infty} |p\rangle\langle p|dp = \mathbb{1} \quad \text{and} \quad \langle x|p\rangle = \frac{1}{\sqrt{2\pi}} e^{ixp}. \tag{3.2}$$

An ideal von Neumann measurement is usually modeled with an instantaneous interaction between the two subsystems, in such a way that one can ignore the individual evolutions of each subsystem compared to the interaction described by the Hamiltonian:

$$H_{int}(t) = \lambda\delta(t - t_0)O \otimes P, \tag{3.3}$$

where λ is the parameter that describes the intensity of the interaction.

Consider the initial state of the composite system given by the product state between the two subsystems in question $|\psi_i\rangle = |\alpha\rangle \otimes |\varphi\rangle$. Thus, the final state will be given by $|\psi_f\rangle = U(t_A, t_B)|\psi_i\rangle \quad (t_A < t_0 < t_B)$, where the unitary operator describing the evolution of this system is given by:

$$U(t_A, t_B) = e^{-i\int_{t_A}^{t_B} H_{int}(t)dt} = e^{-i\lambda O\otimes P}. \tag{3.4}$$

We can expand $|\alpha\rangle$ in the basis formed by the eigenstates of O, getting the following result:

$$|\psi_{(f)}\rangle = e^{-i\lambda O\otimes P}(\sum_j |o_j\rangle\langle o_j|\alpha\rangle \otimes |\varphi\rangle) = \sum_j |o_j\rangle\alpha_j \otimes e^{-i\lambda o_j P}|\varphi\rangle = \sum_j |o_j\rangle \otimes |\varphi_j\rangle\alpha_j, \tag{3.5}$$

such that $|\varphi_j\rangle$ is:

$$|\varphi_j\rangle = e^{-i\lambda o_j P}|\varphi\rangle. \tag{3.6}$$

It is important to note that the states $|\varphi_j\rangle$ form, in general a *non-orthogonal* set which is correlated with the eingestates of O. To analyze the correlation in terms of spacial coordinates, we have the resulting wave function in $\mathcal{H}_M$, by using $(\mathbb{1} \otimes \langle x|)$:

$$(\mathbb{1} \otimes \langle x|)|\psi_f\rangle = \sum_j |o_j\rangle\langle x|V^\dagger_{\lambda o_j}|\varphi_i\rangle\alpha_j, \tag{3.7}$$

where $V_\xi = e^{i\xi P}$ is the family of unit operators that implement the representation of the abelian additive group of translations in the basis of position:

$$V_\xi|x\rangle = |x - \xi\rangle. \tag{3.8}$$

Thus, a correlation is established between the variable to be measured o_j with the continuous variable of position of the measuring particle:

$$(\mathbb{1} \otimes \langle x|)|\psi_f\rangle = \sum_j |o_j\rangle \alpha_j \varphi_i(x - \lambda o_j). \tag{3.9}$$

Note that, in general:

$$\langle \varphi_i| V_{\lambda o_j} |\varphi_i\rangle = \int_{-\infty}^{+\infty} \varphi_i(x - \lambda o_j)\varphi_i^*(x) \neq 0, \tag{3.10}$$

where $\varphi_i^*(x)$ is the complex conjugate of the initial wave function ($\varphi_i(x) = \langle x|\varphi_i\rangle$) of the measuring system constituted by the one-dimensional particle.

A convenient choice for modeling the initial wave-function state of the measuring system is given by the gaussian

$$\varphi_i(x) = (2\pi\Delta^2)^{-\frac{1}{4}} \exp\left(-\frac{x^2}{4\Delta^2}\right) \tag{3.11}$$

so,

$$(\mathbb{1} \otimes \langle x|)|\psi_f\rangle = \sum_j |o_j\rangle \alpha_j (2\pi\Delta^2)^{-\frac{1}{4}} \exp\left(\frac{-(x - \lambda o_j)^2}{4\Delta^2}\right). \tag{3.12}$$

Note that the probability density for the outcome x is:

$$p(x) = \sum_j |\alpha_j|^2 (2\pi\Delta^2)^{-\frac{1}{2}} \exp\left(\frac{-(x - \lambda o_j)^2}{2\Delta^2}\right). \tag{3.13}$$

If Δ is small enough compared to λ, then the gaussian functions in (3.13) do not overlap, we obtain the eigenvalue x with probability $|\alpha_j|^2$ and the meter will be described by one of the "pointer states" of the system, indicating the collapse of the state to be measured to one of the eigenstates of the observable in question. On the other hand, if Δ is large compared to λ, we have the weak measurement regime. In this scenario, we measure an observable without causing an appreciable disturbance, and to this end we must give up our precision in that process, in the sense of interact as little as possible the system with the meter. Although not so widespread initially, weak measurement and weak value gained an important status in the literature over the past 20 years as can be seen in [109, 110, 111, 112]. We are now in position to return on Vaidman's approach to reality.

3.3 WEAK MEASUREMENT

The weak measurement, in the form proposed originally by Aharonov *et.al* can be understood as an infinitesimal interaction between the system to be measured and the measuring system, with the intention of not modifying substantially the state to be measured. So, the weak measurement process can be modeled analogously to the von Neumann, but considering an infinitesimal interaction:

$$H_{int}^{(w)}(t) = \lambda\delta(t - t_0)O \otimes P \qquad (\lambda \to 0). \tag{3.14}$$

Considering $U^{(w)}(t_i, t_f)$ expanded to the first order in λ

$$U^{(w)}(t_i, t_f) \simeq \mathbb{1} - i\lambda O \otimes P, \tag{3.15}$$

we have that:

$$|\Psi_f\rangle = (\mathbb{1} - i\lambda O \otimes P)\, |\Psi_i\rangle, \tag{3.16}$$

with $|\Psi_i\rangle = |\alpha\rangle \otimes |\varphi_i\rangle$ given by the tensor product of the state to be measured with the metering state. As shown in [85], the above evolution does not substantially modify the state, in a way that, when $\lambda \to 0$, $|\Psi_f\rangle \to |\Psi_i\rangle$.

The point is, if $|\Delta|$ is taken to be very greater than λ multiplied by the eigenvalues of the observable that is being measured, we have that the probability distribution

$$p(x) = \sum_j |\alpha_j|^2 \,(2\pi\Delta^2)^{-\frac{1}{2}} \exp\left(-\frac{(x - \lambda o_j)^2}{2\Delta^2}\right), \tag{3.17}$$

can be written as

$$p(x) \approx (2\pi\Delta^2)^{-\frac{1}{2}} \exp\left(-\frac{(x - \lambda \sum_j |\alpha_j|^2 o_j)^2}{2\Delta^2}\right). \tag{3.18}$$

Note that due to the weak measurement, the center of the gaussian distribution is displaced by a quantity equal to the expectation value of the observable O, given by $\sum_j |\alpha_j|^2 o_j$. It is important to discuss now what the *effective* result is for a weak measurement for pre and post-selected states, an aspect that defines the *two state vector formalism* (TSVF) of QM. As originally presented by Aharonov, Bergmann, and Lebowitz in 1964 [29] the TSVF is actually a *reformulation* of QM in terms of pre and post-selected time boundary conditions for the evolution of quantum systems. Since the TSVF is a reformulation and *not* a modification of the standard quantum formalism, it is not possible to distinguish experimentally one from another. In what follows, we discuss the consequences of assuming such a boundary temporal condition.

3.3.1 TSVF, weak values, and weak reality

We now want to calculate the conditional probability of a measurement of an observable $O = \sum_j |o_j\rangle o_j \langle o_j|$ at the intermediary instant t for respective non-orthogonal pre- and post-selected states $|\alpha_{in}\rangle$ and $|\beta_{fin}\rangle$, respectively, obtained at instants t_{in} and t_{fin} such that $t_{in} < t < t_{fin}$. The probability of an event A conditioned to the result of an event B is given by the relative probability

$$P\left(A|\, B\right) = \frac{P(A,B)}{P(B)}, \tag{3.19}$$

where $P(A,B)$ is the probability of happening *both* events A and B, and $P(B)$ is the probability of occurrence only of the event B. The event A we take as being the measure of the observable O at the instant t with pre and post-selection $|\alpha_{in}\rangle$ and $|\beta_{fin}\rangle$, resulting in a specific o_j, while the event B represents the fact that we measured O in the instant t (with any result) given the states $|\alpha_{in}\rangle$ and $|\beta_{fin}\rangle$. The probability amplitude for the event A is given by $\langle\beta_{fin}|\, U_{t\to t_{fin}}\,|o_j\rangle\,\langle o_j|\, U_{t_{in}\to t}\,|\alpha_{in}\rangle$. In this way, applying 3.19, we get the formula developed by ABL [29]:

$$p_{ABL}(o_j, t|\alpha_{in}, t_{in}; \beta_{fin}, t_{fin}) = \frac{\left|\langle\beta_{fin}|\, U_{t\to t_{fin}}\,|o_j\rangle\,\langle o_j|\, U_{t_{in}\to t}\,|\alpha_{in}\rangle\right|^2}{\sum\limits_j \left|\langle\beta_{fin}|\, U_{t\to t_{fin}}\,|o_j\rangle\,\langle o_j|\, U_{t_{in}\to t}\,|\alpha_{in}\rangle\right|^2}. \tag{3.20}$$

Note that the above expression is temporally symmetric. Following Aharonov and collaborators [31], this formula can be interpreted in the following way. We start with the pre-selected state $|\alpha_{in}\rangle$. This state evolves with $U_{t_{in}\to t} = \exp(-iH(t - t_{in}))$ with H representing the Hamiltonian of the system. So, the term $|\langle o_j| U_{t_{in}\to t} |\alpha_{in}\rangle|^2$ represents the probability of having $|o_j\rangle$ in a measurement of the observable O in a time t for the state $U_{t_{in}\to t} |\alpha_{in}\rangle$. Analogously the state $|o_j\rangle$ evolves toward $U_{t\to t_{fin}}$ and $\left|\langle \beta_{fin}| U_{t\to t_{fin}} |o_j\rangle\right|^2$ provides the probability of obtaining $|\beta_{fin}\rangle$. Given the three stages, the *conditional probability* of having $|o_j\rangle$ at the intermediary time $t \in [t_{in}, t_{fin}]$ between the pre and post-selection is given by the product of the two previously explained expressions with proper normalization, that is, summing up all possible events conditioned to a pre-selection $|\alpha_{in}\rangle$ at t_{in} and a post-selection $|\beta_{fin}\rangle$ at t_{fin}. As the operator of temporal evolution is unitary, we have that

$$(U_{t\to t_{fin}})^\dagger = \exp(-iH(t_{fin} - t))^\dagger = \exp(-iH(t - t_{fin})) = U_{t_{fin}\to t}. \tag{3.21}$$

Using that, the TSVF considers that a quantum system at a given instant t is completely described by a bi-vector, as follows:

$$\langle \beta(t)| \otimes |\alpha(t)\rangle\,, \tag{3.22}$$

where $|\alpha(t)\rangle$ is a state that acts on $\mathcal{H}$, and is given by the result of a strong measurement carried out at $t_{in} < t$ (pre-selection) which evolved "forward in time" (*causal evolution*) through a given by unitary operation

$$|\alpha(t)\rangle = U(t_1, t)\,|\alpha(t_1)\rangle\,, \tag{3.23}$$

and $\langle \beta(t)|$ belongs to the dual space $\overline{\mathcal{H}}$, and is given by the result of a strong measurement carried out at $t_{fin} > t$ (post-selection) that "evolved backwards in time" (*retrocausal evolution*) through a given unitary operation

$$\langle \beta(t)| = \langle \beta(t_2)|\,\hat{U}^\dagger(t, t_2). \tag{3.24}$$

Note that the expression 3.20 is symmetrical under change of pre and post-selected states. This is, the probability of obtaing o_j do not modify if we change t for $-t$. When we consider an ensemble with a large number of systems all prepared with the same pre-selected state, the two-state formalism predicts that this ensemble can be *a priori* subdivided into sub-ensembles, each with different post-selected states. This means that a complete description of an ensemble in this formalism depends on "future information " including a future decision on what measurement will be taken and which is the result of this future collapse of the wave-function. Therefore, there are retro-causal elements in this formulation (which maintains a symmetry between the future and the past), since the system "carries" information about its future destination, although this information is inaccessible to any outside observer, thus been considered as a type of a temporal hidden variable.

Guaranteed that the weak measurement will not cause disturbance (at an ideal limit), what happens when we condition a weak measurement to a future strong measurement (post-selection)? It is important to note that as we are making a conditioning to a future result, the results of these measurements have be to seen as an effective interaction (we refer the reader to [85] for further details). We have

$$|\Psi_f\rangle = ((\langle\beta| \otimes \mathbb{1})U^{(w)}(t_i, t_f)(|\alpha\rangle \otimes |\varphi_{(i)}\rangle), \tag{3.25}$$

where $|\alpha\rangle$ and $|\beta\rangle$ are respectively the pre and post-selected states of the subsystem to be measured. Consider again, $U^{(w)}(t_i, t_f)$ expanded to the first order in λ, it follows that:

$$|\Psi_f\rangle \simeq ((\langle\beta| \otimes \mathbb{1})(\mathbb{1} - i\lambda O \otimes P)(|\alpha\rangle \otimes |\varphi_i\rangle), \tag{3.26}$$

$$|\Psi_f\rangle \simeq \langle\beta|\alpha\rangle(\mathbb{1} - i\lambda\frac{\langle\beta|O|\alpha\rangle}{\langle\beta|\alpha\rangle}P)|\varphi_i\rangle, \tag{3.27}$$

$$|\Psi_f\rangle \simeq \langle\beta|\alpha\rangle(\mathbb{1} - i\lambda O_w P)|\varphi_i\rangle \qquad \text{with} \qquad O_w = \frac{\langle\beta|O|\alpha\rangle}{\langle\beta|\alpha\rangle}, \tag{3.28}$$

$$|\Psi_f\rangle \simeq \langle\beta|\alpha\rangle e^{-i\lambda O_w P}|\varphi_i\rangle. \tag{3.29}$$

The quantity O_w is recognized as the *weak value* of the observable O given this pre and post selection. Also, it can assume values throughout the complex plane, thus being different from the real expectation value that we have in the usual formalism. Both quantities are related as follows:

$$\langle O \rangle_{|\alpha\rangle} = \langle\alpha|\, O|\alpha\rangle = \langle\alpha| \sum_{\beta} |\beta\rangle\langle\beta|O|\alpha\rangle = \sum_{\beta} |\langle\beta|\alpha\rangle|^2 \, \frac{\langle\beta|O|\alpha\rangle}{\langle\beta|\alpha\rangle}, \tag{3.30}$$

With that, Vaidman extends the discussion of physical reality to the context of weak measurements. He suggests that one advantage of defining elements of reality from weak measurements, is that it is possible to encompass all the results provided by projective measurements whose result is an eigenvalue of the observable being measured and, in addition, we have the expected value of the observable and the weak value as elements of reality as well.

It is important to note that weak measurement and weak values are not only discussed in the context of TSVF. For example, another relevant aspect of weak measurements were discussed by Oreshkov and Brun [87]. It is well known that any unitary transformation can be implemented as a sequence of weak (i.e., infinitesimal) unitary transformations. Oreshkov and Brun asked if a similar decomposition exists for generalized measurements. Such decomposition would allow us to think of POVMs as resulting from continuous stochastic evolutions. They constructed an operator that has the structure of a random walk along a curve in the state space, with the measurement ending when one of the end points is reached. This shows that any measurement can be generated by weak measurements, and hence that weak measurements are universal. The weak-measurement dichotomic operators introduced by Oreshkov and Brun [87] is

$$P_{\pm}(x) = \sqrt{\frac{1 \mp \tanh x}{2}}\Pi_0 + \sqrt{\frac{1 \pm \tanh x}{2}}\Pi_1, \tag{3.31}$$

with $x \in \mathbb{R}$ and $\Pi_0 + \Pi_1 = P_+^2 + P_-^2 = \mathbb{1}$ for projectors Π_0 and Π_1 acting on $\mathcal{H}_{\mathcal{B}}$. We have that the projection of these operators gives the following post-measurement state

$$\rho_{\mathcal{A}|P_{\pm}} = \mathrm{Tr}_{\mathcal{B}}[(\mathbb{1} \otimes P_{\pm})\rho(\mathbb{1} \otimes P_{\pm})]/p_{\pm}, \tag{3.32}$$

with probabilities

$$p_{\pm} = \mathrm{Tr}[(\mathbb{1} \otimes P_{\pm})\rho(\mathbb{1} \otimes P_{\pm})]. \tag{3.33}$$

We will return later on to these operators to discuss the notion of quantum discord based on weak measurements. Before that, as we are interested in this work in investigating fundamental aspects related to the measurement process in the QM, we will discuss in more detail the main problems regarding the interpretation of the measurement process in QM, this is, when and how the final state emerges out of all possible measurement results and present the version of the decoherence program to explain these questions.

3.4 MEASUREMENT PROBLEM

Consider a "typical" dynamics corresponding to a von Neumann measurement, we can summarize it as follows:

$$|\Psi_I\rangle = |o_j\rangle \otimes |\varphi\rangle \overset{measurement}{\longrightarrow} |\Psi_F\rangle = |o_j\rangle \otimes |\varphi_j\rangle\,, \tag{3.34}$$

where $|\varphi\rangle$ is an arbitrary state in $\mathcal{H}_M^{(m)}$. In this case, we have a non-entangled product state. Assuming that the physical system now begins in a superposition $|\alpha\rangle = \sum_j |o_j\rangle\,\alpha_j$, then, by linearity, the dynamics corresponding to the interaction between the physical system and the apparatus is

$$|\alpha\rangle \otimes |\psi\rangle = \sum_j |o_j\rangle\,\alpha_j \otimes |\varphi\rangle \overset{measurement}{\longrightarrow} \sum_j |o_j\rangle \otimes |\varphi_j\rangle\,\alpha_j, \tag{3.35}$$

which is a generically entangled state of the system $\mathcal{H}_S^{(n)} \otimes \mathcal{H}_M^{(m)}$. Note that the superposition contained only in the system to be measured $\mathcal{H}_S^{(m)}$ was extended via *entanglement* to the joint system, correlating the states $|o_j\rangle$ and $|\varphi_j\rangle$. This situation is called von Neumann's ideal pre-measurement in the literature. The measurement itself happens when (in some way) a particular element of the combination $|o_j\rangle \otimes |\varphi_j\rangle\,\alpha_j$ "materializes" via the observation of an element $|\varphi_j\rangle$ of the apparatus with a probability $P_j = |\alpha_j|^2$.

In this scenario, the so-called *measurement problem* emerges, which consists basically in answering the following questions. What makes the measurement process special to the point of being necessary to create a postulate for this process? When and why only one state materializes if we have a combination of various possibles results? By carefully examining the attempts to explain the measurement process in QM, we can say that during the last century, several proposals were made with the intention of formalizing this question in a satisfactory way. It is natural that we can found several different formulations for the measurement problem, since the notion of what would be satisfactory varies from interpretation to interpretation. Even in the same interpretation there were created many different versions (reformulations) of the measurement problem, as can be seen in [14], where one finds an example of five different formulations for the measurement problem using only the standard interpretation of QM. We can say that there is no consensus in the scientific community of what is the correct axiomatic formulation to deal with the measurement process in order to satisfy our physical interpretation. An approach to the measurement problem usually adopted by many physicists is the decoherence paradigm, being announced basically in terms of two questions. The first consists of to explain how the quantum-classical transition occurs in a measurement process, this includes the preferred basis problem and addressing the question of why classical macroscopic states do not interfere as a quantum superposed state. The second concerns the discussion of how the specific results is obtained in reducing the quantum state, that is, how specific results emerge from the intrinsically random original overlap.

More precisely speaking, the first problem stems from the fact that we can write the resulting state that describes a pre-measurement in an infinity of ways, generating an ambiguity,

$$|\Psi_F\rangle = \sum_a |\varphi_a\rangle \otimes |\psi_a\rangle \, \varphi_a = \sum_b |\varphi\prime_b\rangle \otimes |\psi\prime_b\rangle \, \varphi\prime_b = \sum_c |\varphi"_c\rangle \otimes |\psi"_c\rangle \, \varphi"_c = ... \qquad (3.36)$$

To see this, we simply insert a resolution of identity with arbitrary orthonormal basis $\{|\varphi\prime_b\rangle\}, \{|\varphi\prime_c\rangle\}, ...$ in the following way:

$$|\Psi_F\rangle = \sum_a |\varphi_a\rangle \otimes |\psi_a\rangle \, \varphi_a = \sum_{a,b} |\varphi\prime_b\rangle \langle\varphi\prime_b|\varphi_a\rangle \otimes |\psi_a\rangle \, \varphi_a = \sum_b |\varphi\prime_b\rangle \otimes |\psi\prime_b\rangle \, \varphi\prime_b, \qquad (3.37)$$

with

$$|\psi\prime_b\rangle\,\varphi\prime_b = \sum_a |\psi_a\rangle\,\langle\varphi\prime_b|\varphi_a\rangle\,\varphi_a, \tag{3.38}$$

The question is, why do we perceive the macroscopic systems in a specific set of values, such as a well-defined spatial position, if these macroscopic systems *a priori* would also be subject to the laws of QM? Is there a border or cut separating the "quantum world" and the "classical world"? These questions were intensively debated throughout the 1980s [15, 16, 17]. It can be observed in the literature that many physicists consider that the decoherence program gives prominent rules to explain the measurement problem. In considering the measurement model as a triple quantum system (system to be measured, measuring device, and environment) it is possible to argue and demonstrate via toy-models that the terms referring to a certain measure of quantum superposition between the system to be measured with the measuring device rapidly goes to zero in its local form, that is, the coherence of quantum states is maintained in the system as a whole but is suppressed locally when the environmental is traced out, as we shall discuss next through the simple spins-bath toy model introduced by Zurek.

3.4.1 Decoherence paradigm

This model was introduced by Zurek [17], and consists of a system formed by a particle of spin 1/2 (one qubit), with two possible states $|0\rangle$ and $|1\rangle$ (spin up and down, respectively), interacting with an environment consisting of N particles of spin 1/2 (N qubits). The effect of the unitary evolution due to the Hamiltonian of each individual subsystem can be neglected in view of the interaction. This can be modeled in the following form:

$$H \approx H_{int} = -\frac{1}{2}\sigma_z \otimes \sum_{k=1}^{N} \mathbb{1}_1 \otimes \mathbb{1}_2 \otimes ... \otimes \mathbb{1}_{k-1} \otimes [g_k\sigma_z^{(k)}] \otimes \mathbb{1}_{k+1} \otimes ... \otimes \mathbb{1}_N \tag{3.39}$$

The above sum represents the fact that the particles of the environment do not interact with each other, the coefficients g_k represent the intensity of interaction and $\sigma_z^{(k)}$ is one of the Pauli Matrices $\sigma_z^{(k)} = |0_k\rangle\,\langle 0_k| - |1_k\rangle\,\langle 1_k|$

The state of the combined system before the interaction is:

$$|\Psi(0)\rangle = (a\,|0\rangle + b\,|1\rangle)\bigotimes_{k=1}^{N}(\alpha_k\,|0_k\rangle + \beta_k\,|1_k\rangle). \tag{3.40}$$

The dynamics is governed by a unitary time evolution operator which yields (considering units such that $\hbar = 1$):

$$|\Psi(t)\rangle = a\,|0\rangle\,|E_0(t)\rangle + b\,|1\rangle\,|E_1(t)\rangle\,, \tag{3.41}$$

where

$$|E_0(t)\rangle = |E_1(-t)\rangle = \bigotimes_{k=1}^{N}\left[\exp\left(\frac{ig_k t}{2}\right)\alpha_k\,|0_k\rangle + \exp\left(\frac{-ig_k t}{2}\right)\beta_k\,|1_k\rangle\right]. \tag{3.42}$$

Taking the partial trace in the environment system gives

$$Tr_{env}[\rho(t)] = |a|^2\,|0\rangle\,\langle 0| + |b|^2\,|1\rangle\,\langle 1| + r(t)ab^*\,|0\rangle\,\langle 1| + r^*(t)a^*b\,|1\rangle\,\langle 0|\,, \tag{3.43}$$

with

$$r(t) = \langle E_1(t)|E_0(t)\rangle = \prod_{k=1}^{N}(|\alpha_k|^2\exp(ig_k t) + |\beta_k|^2\exp(-ig_k t)). \tag{3.44}$$

The function $r(t)$ is called decoherence factor. The point is that besides being a periodic function, the factor $|r(t)|^2$ quickly goes to zero when N increases [17]. Also, in more realistic models, for which there is no trivial frequency distribution g_k, the period of this function can become comparable to the lifetime of the universe, indicating that the periodicity is excluded to all practical proposes. This shows that the coherence locally disappears while it is retained in the system as a whole. So, the non-diagonal terms related to quantum interference vanishes very quickly when we take the partial trace over the environment. We can also easily identify states that are not affected by decoherence interaction, these states are robust because they remain unchanged under the decoherence process and therefore were crucial for our further understanding of the quantum-classical transition. In fact, it is accepted that the position of the center of mass of macroscopic bodies is well defined because the fundamental Hamiltonian interactions are functions of position. Also, one can invoke the notion of Quantum Darwinism and use the notion of redundancy to justify the objectivity in a measurement process. This justification assumes that as the interactions of a quantum system with a very large number of subsystems in the environment generate many redundant copies of information about its pointer state. The environment plays the role of a witness, acting effectively as a natural selector (*einselection* [41]). For many physicists, this suffices to explain the pointer basis problem and the problem of not observing quantum interference at macroscopic levels.

There are interpretations (for example, in the Many-Worlds interpretation MWI, where in each "universe" we have a story regarding one of the possible states of the global superposition) in which the collapse of the wave-function is not a fundamental element of the theory, that is, the reduction postulate in these interpretations is seen only as an apparent phenomenon. The interpretation of the measurement process when we have only one observer is already a delicate issue, however, as well emphasized Everett [126], the situation may become quite paradoxical if we allow the existence of more than one observer. Also, since the collapse is an apparent phenomenon, how to justify the fact that we experience in the laboratory only one branch among all possible? This is another controversial point because to answer these questions some physicists have to incorporate the notion of a "conscious observer", arguing that this entity is also correlated with the experiment, being therefore part of the overall wave function as will be discussed in the next example. Although controversial, there are many current advocates of the MWI [114],

> "For me Bell's result was the first reason to accept the MWI. Since then, the discovery of teleportation and of the interaction-free measurements turned my belief into a strong conviction".

On the sequence, we explore Everett's paradox in a more detailed scenario.

3.4.2 Everett's paradox and MWI

Suppose that during the period of a day, it is assigned to an experimental physicist Alice, the task of performing measurements on a quantum system, in a laboratory considered to be ideally isolated, and from this measurement Alice will write down her result in a notebook. For simplicity, let's assume that the quantum system in question is an electron and the physical property to be measured is its spin in a certain direction. Alice then prepares her measuring device, a Stern-Gerlach apparatus. In addition, let us consider that a theoretical physicist Bob, external to the lab, is assigned the task of computing the total resulting state of the systems present in Alice's laboratory for the end of the experiment, considering that Bob knows the observable that Alice will measure, and the initial states of each system present in the laboratory. Also, it is ensured that the lab has remained closed the whole day of the experiment.

The question is, can we, in this case, treat Bob's description as well as von Neumann treated the measurement process in QM? As Everett argues, if we want to deny the possibility of Bob using QM to describe this process, then we must provide some alternative description for systems containing observers or measuring apparatus. In addition, we would have to have a criterion to say precisely what kind of systems would have this preferred positions of "measurement apparatus" or "observes".

The temporal evolution of the spin of the electron, through a known closed dynamics would produce, according to the classic Laplacian view, a new, well-determined state of spin (an element of reality for the magnetic dipole moment of the electron). QM theory however, with the *superposition principle*, predicts that under a unitary evolution U, the state will generally evolve towards a superposition of the eigenstates of the observable in question, so that an unambiguous discrimination of an element of reality for the spin, as would like the Laplacian determinism, is denied. Thus, unlike classical theory, the closed evolution of a well-defined property does not guarantee determinism.

From Bob's viewpoint (the outsider), the measurement process may, in principle, be governed by physical interactions and be described by a unitary dynamics, with superpositions now involving the physical system, the measure device and Alice. On the other hand, from Alice's perspective, after a measurement the reality of the spin is established and recorded by means of its annotation and memory. How then, to justify this difference between what an observer is computing and what another observer recorded as an effective result of the measurement? Would be the theoretical description of Bob incomplete or even wrong or can we argue that based on this mental experiment we can not associate, according to the laws of QM, an objective reality for Alice?

The general objective of our work is to investigate the measurement problem and the Everett paradox by taking an informational point of view, looking for connections with the discussion of reality, weak and strong measurements and quantum correlations. To this end, we will introduce the quantifiers of information [108] and reality [53] which pave the way to present our own results.

3.5 INFORMATION AND PHYSICAL REALITY

Some works have argued that a better understanding of the physical nature can be achieved through the concept of information. Bruckner and Zeilinger defend that quantum physics is an elementary theory of information [104, 105] and that even though information should not be taken as replacing the notion of reality, in their approach "the notions of reality and of information are on equal footing" [106], which suggests some ontological status for information. The information interpretations of QM can be considered as modern information-theoretic versions of the orthodox Copenhagen interpretation, thus in connection with Bohr's ideas [9]. This viewpoint is close to Quantum Bayesianism (Qbism), which is a reconstruction of QM that mixes subjective elements, associated with the probabilistic information that an agent has about the world, with objective elements, which are identified as the Hilbert space dimension of the quantum systems: "Dimension is something a body holds by itself, regardless of what an agent thinks of it" [107]. For an overview of conceptions of information and reality in physics we refer the reader to [9] and [95]. Next, we define what we will use as a quantifier of information.

3.5.1 Information

The von Neumann entropy $S(\rho)$ is the quantum-mechanical object widely used to deal with the amount of information associated with the quantum state ρ. Here, however, we follow the approach of Ref. [108] and define the amount of information associated with a generic quantum state ρ in a Hilbert space $\mathcal{H}$ of dimension d as

$$I(\rho) := \ln d - S(\rho). \tag{3.45}$$

We interpret this as the amount of information available in the reference frame where ρ has been prepared. Clearly, I is maximum (minimum) for a pure (totally mixed) state. In the present approach, therefore, $S(\rho)$ quantifies the ignorance about the state ρ.

Consider now a bipartition such that $\mathcal{H} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$ and $d = d_{\mathcal{A}} d_{\mathcal{B}} = \dim \mathcal{H}$. It is straightforward to show that

$$I(\rho) = I(\rho_{\mathcal{A}}) + I(\rho_{\mathcal{B}}) + I_{\mathcal{A}:\mathcal{B}}(\rho), \tag{3.46}$$

where $I(\rho_{\mathcal{A}(\mathcal{B})})$ is the information related to the subsystem $\mathcal{A}(\mathcal{B})$,

$$I_{\mathcal{A}:\mathcal{B}}(\rho) = S(\rho_{\mathcal{A}}) + S(\rho_{\mathcal{B}}) - S(\rho) \tag{3.47}$$

is the mutual information, and

$$\rho_{\mathcal{A}(\mathcal{B})} = \mathrm{Tr}_{\mathcal{B}(\mathcal{A})} \rho \tag{3.48}$$

is the reduced state. The above relation shows that the total information is the sum of local and nonlocal terms, that is, part of the total information is related to the individual subsystems and part is shared by them. The latter term ($I_{\mathcal{A}:\mathcal{B}}$), which is also a measure of the total correlations between $\mathcal{A}$ and $\mathcal{B}$, quantifies the information that $\mathcal{A}$ has about $\mathcal{B}$, and vice-versa. Most importantly, we can check, via unitary invariance of the von Neumann entropy, that in closed systems the total available information is constant, that is $\Delta I = 0$. As shown in Ref. [108], this conservation law allows us to speak of an information flow.

For example, when a two-qubit state

$$|\psi_0\rangle = |a_0\rangle \left(|b_1\rangle + |b_2\rangle\right) / \sqrt{2} \tag{3.49}$$

evolves to

$$|\psi_t\rangle = \left(|a_1\rangle|b_1\rangle + |a_2\rangle|b_2\rangle\right)/\sqrt{2}, \tag{3.50}$$

with

$$\langle a_i|a_j\rangle = \langle b_i|b_j\rangle = \delta_{ij}, \tag{3.51}$$

the total information $I = 2\ln 2$, which initially manifested exclusively as local information, is fully transformed into shared information (in this case, entanglement).

Taking

$$S_{\mathcal{A}|\mathcal{B}}(\rho) = S(\rho) - S(\rho_{\mathcal{B}}) \tag{3.52}$$

as the definition for the conditional quantum entropy (the entropy of $\mathcal{A}$ given information about $\mathcal{B}$) and introducing

$$I_{\mathcal{A}|\mathcal{B}}(\rho) = \ln d_{\mathcal{A}} - S_{\mathcal{A}|\mathcal{B}}(\rho) \tag{3.53}$$

as the conditional information, we can rewrite Eq. (3.46) in the form

$$I(\rho) = I(\rho_{\mathcal{B}}) + I_{\mathcal{A}|\mathcal{B}}(\rho), \tag{3.54}$$

which is particularly interesting for instances where only the part $\mathcal{B}$ can be accessed. Next, we define what we will use as a quantifier of reality [53].

3.5.2 A measure of physical reality

Throughout this thesis, we employ a notion of reality that has recently been introduced by Bilobran and Angelo (BA) [53]. Its main advantage is that it is quantitative and operational. BA consider a preparation ρ on $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$ submitted to a protocol of unrevealed measurements of a generic observable $A = \sum_a a A_a$, with projectors $A_a = |a\rangle\langle a|$, acting on $\mathcal{H}_{\mathcal{A}}$. Since the outcome of the measurement is kept secret, the resulting state reads

$$\Phi_A(\rho) = \sum_a \left(A_a \otimes \mathbb{1}_{\mathcal{B}}\right) \rho \left(A_a \otimes \mathbb{1}_{\mathcal{B}}\right) = \sum_a p_a A_a \otimes \rho_{\mathcal{B}|a}, \tag{3.55}$$

where $\rho_{\mathcal{B}|a} = \langle a|\rho|a\rangle/p_a$ and $p_a = \mathrm{Tr}[(A_a \otimes \mathbb{1}_{\mathcal{B}})\rho]$. Under the premise that a measurement establishes the reality of an observable, BA propose to take $\Phi_A(\rho)$ as a state of reality for

A and $\rho = \Phi_A(\rho)$ as a formal criterion of reality. With that we can compute the degree of irreality of the observable A given the preparation ρ as

$$\mathfrak{I}(A|\rho) := S(\Phi_A(\rho)) - S(\rho), \tag{3.56}$$

where $S(\rho) = -\mathrm{Tr}(\rho \ln \rho)$ stands for the von Neumann entropy. The above formula can be viewed as an entropic distance between the state ρ under scrutiny and the state of reality $\Phi_A(\rho)$. This quantifier is non-negative and vanishes if and only if $\rho = \Phi_A(\rho)$. Also, we see that the following decomposition holds. Consider the nonminimized version of the one-way quantum discord developed in [72]:

$$\begin{aligned}
D_A(\rho) =& I_{\mathcal{A}:\mathcal{B}}(\rho) - I_{\mathcal{A}:\mathcal{B}}(\Phi_A(\rho)) && (3.57)\\
=& S(\rho_{\mathcal{A}}) + S(\rho_{\mathcal{B}}) - S(\rho) - S(\Phi_A(\rho_{\mathcal{A}})) - S(\rho_{\mathcal{B}}) + S(\Phi_A(\rho)) && (3.58)\\
=& S(\Phi_A(\rho)) - S(\rho) - (S(\Phi_A(\rho_{\mathcal{A}})) - S(\rho_{\mathcal{A}}))\\
=& \mathfrak{I}(A|\rho) - \mathfrak{I}(A|\rho_{\mathcal{A}})
\end{aligned}$$

In this formulation,

$$\mathfrak{I}(A|\rho) = \mathfrak{I}(A|\rho_{\mathcal{A}}) + D_A(\rho), \tag{3.59}$$

it is noticeable that the irreality of A is the sum of the local irreality (that is, the irreality of A given the reduced state $\rho_{\mathcal{A}}$) with quantum correlations associated with measurements of A.

From now on we enter on the original part of the thesis, published in [36] and [64].

4 INFORMATION-REALITY COMPLEMENTARITY

The results and most of the content of this chapter are published in [P. R. Dieguez and R. M. Angelo, Phys. Rev. A **97**, 022107 (2018)] [36]. This work was selected as Editor's Suggestion in PRA. Also it was selected for the Brazilian Physical Society (SBF) [116] and Federal University of Ouro Preto (UFOP) [117] newsletters.

4.1 STRONG AND WEAK MEASUREMENTS

One of the basic postulates of QM is the state reduction (collapse). It clearly is an effective theoretical tool, a prescription for obtaining the state resulting from a measurement without in any way accounting for the details of the physical interaction with the measurement apparatus. As such, there is no reason *a priori* to view the collapse as a real physical phenomenon emerging from the dynamics between the system and the apparatus. In this section we employ this formal perspective. Consider a preparation ρ on $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$ ($\dim \mathcal{H}_{\mathcal{A},\mathcal{B}} = d_{\mathcal{A},\mathcal{B}}$). According to the quantum axioms, if a nondegenerate-spectrum operator $A = \sum_a a A_a$, with projectors $A_a = |a\rangle\langle a|$, is measured in a given run of the experiment and a result a is obtained, then the resulting state is given by

$$\mathcal{C}_{a|A}(\rho) := \frac{(A_a \otimes \mathbb{1}_{\mathcal{B}})\, \rho\, (A_a \otimes \mathbb{1}_{\mathcal{B}})}{\mathrm{Tr}\left[(A_a \otimes \mathbb{1}_{\mathcal{B}})\, \rho\, (A_a \otimes \mathbb{1}_{\mathcal{B}})\right]} = A_a \otimes \rho_{\mathcal{B}|a}. \tag{4.1}$$

Here $\mathcal{C}_{a|A}$ is a linear map that formally describes the collapse of the state vector. After a projective measurement of this type, the observer is granted with full information about the reduced state ($\rho_{\mathcal{A}} = A_a$) of the system. In fact, after the measurement the information about the subsystem $\mathcal{A}$ reaches its maximum value $I_{\mathcal{A}} = \ln d_{\mathcal{A}}$. Notice that $\mathcal{C}^n_{a|A}(\rho) = \mathcal{C}_{a|A}(\rho)$ for $n \geqslant 1 \in \mathbb{Z}$, which correctly implements the condition of repeatability of projective measurements. In addition, we have that $\mathcal{C}_{a'|A}\mathcal{C}_{a|A}(\rho) = 0$ and $\mathcal{C}_{a'|A'}\mathcal{C}_{a|A}(\rho) = \mathcal{C}_{a'|A'}(\rho)$ for generic (eventually incompatible) observables A and A' acting on $\mathcal{H}_{\mathcal{A}}$.

We now devise a map that allows us to effectively interpolate between weak and projective measurements. We assume that under the probing process the state ρ is led to

$$\mathcal{C}^{\epsilon}_{a|A}(\rho) := (1-\epsilon)\,\rho + \epsilon\,\mathcal{C}_{a|A}(\rho), \tag{4.2}$$

with $\epsilon \in (0,1)$. It is clear that $\mathcal{C}^{\epsilon}_{a|A}$ represents a strong projective measurement for $\epsilon \to 1$ and no measurement at all for $\epsilon \to 0$. For small ϵ the map implies just a slight change in the preparation ρ, thus suitably simulating the notion of a weak measurement. Several properties can be derived for the map (4.2). First, for $\{A, A'\}$ acting on $\mathcal{H}_{\mathcal{A}}$ and B acting on $\mathcal{H}_{\mathcal{B}}$ one has that $[\mathcal{C}^{\epsilon}_{a|A}(\rho), \mathcal{C}^{\delta}_{a'|A'}(\rho)] \neq 0$ and $[\mathcal{C}^{\epsilon}_{a|A}(\rho), \mathcal{C}^{\delta}_{b|B}(\rho)] = 0$. Second, we note that because $\mathrm{Tr}\rho = \mathrm{Tr}\mathcal{C}_{a|A}(\rho) = 1$, the map $\mathcal{C}^{\epsilon}_{a|A}$ preserves the trace. Another point to note is that the map preserve hermiticity and positivity. Because $\mathcal{C}_{a|A}(\rho)$ is linear, the map $\mathcal{C}^{\epsilon}_{a|A}$ is also linear, i.e., $\mathcal{C}^{\epsilon}_{a|A}(\rho_1+\rho_2) = \mathcal{C}^{\epsilon}_{a|A}(\rho_1) + \mathcal{C}^{\epsilon}_{a|A}(\rho_2)$. Using these properties, we can study successive applications of the map. It follows that:

$$\begin{aligned}
(\mathcal{C}^{\epsilon}_{a|A})^2(\rho) &= (1-\epsilon)(\mathcal{C}^{\epsilon}_{a|A})(\rho) + \epsilon\mathcal{C}_{a|A}(\rho) && (4.3)\\
(\mathcal{C}^{\epsilon}_{a|A})^3(\rho) &= (1-\epsilon)(\mathcal{C}^{\epsilon}_{a|A})^2(\rho) + \epsilon\mathcal{C}_{a|A}(\rho) &&\\
&\vdots &&\\
(\mathcal{C}^{\epsilon}_{a|A})^n(\rho) &= (1-\epsilon)(\mathcal{C}^{\epsilon}_{a|A})^{n-1}(\rho) + \epsilon\mathcal{C}_{a|A}(\rho). && (4.4)
\end{aligned}$$

Using this recursive relation, we obtain

$$\begin{aligned}
(\mathcal{C}^{\epsilon}_{a|A})^n(\rho) &= (1-\epsilon)(\mathcal{C}^{\epsilon}_{a|A})^{n-1}(\rho) + \epsilon\mathcal{C}_{a|A}(\rho) && (4.5)\\
&= (1-\epsilon)^2(\mathcal{C}^{\epsilon}_{a|A})^{n-2}(\rho) + \epsilon\mathcal{C}_{a|A}(\rho)[(1-\epsilon)+1] &&\\
&= (1-\epsilon)^3(\mathcal{C}^{\epsilon}_{a|A})^{n-3}(\rho) + \epsilon\mathcal{C}_{a|A}(\rho)[(1-\epsilon)^2+(1-\epsilon)+1] &&\\
&\vdots &&\\
(\mathcal{C}^{\epsilon}_{a|A})^n(\rho) &= (1-\epsilon)^p(\mathcal{C}^{\epsilon}_{a|A})^{n-p}(\rho) + \epsilon\mathcal{C}_{a|A}(\rho)\left[\frac{1-(1-\epsilon)^p}{\epsilon}\right], && (4.6)
\end{aligned}$$

where we used

$$\sum_{k=0}^{m-1} a^k = \frac{1-a^m}{1-a}. \tag{4.7}$$

Making $p = n$, we finally obtain

$$(\mathcal{C}^{\epsilon}_{a|A})^n(\rho) = \mathcal{C}_{a|A}(\rho) + (1-\epsilon)^n\big[\rho - \mathcal{C}_{a|A}(\rho)\big] = (1-\epsilon)^n\rho + \big[1-(1-\epsilon)^n\big]\mathcal{C}_{a|A}(\rho). \quad (4.8)$$

Notice that $(\mathcal{C}^{\epsilon\to 1}_{a|A})^n(\rho) = \mathcal{C}_{a|A}(\rho)$, as expected. Also, one has that $(\mathcal{C}^{\epsilon}_{a|A})^n(\rho) = (\mathcal{C}^{1-(1-\epsilon)^n}_{a|A})(\rho)$, which shows that n weak measurements of intensity ϵ is equivalent to a single weak measurement of intensity $1-(1-\epsilon)^n$. Finally,

$$\lim_{n\to\infty}(\mathcal{C}^{\epsilon}_{a|A})^n = \mathcal{C}_{a|A}, \quad (4.9)$$

which means that infinitely many weak measurements, executed either sequentially or simultaneously, is equivalent to a projective measurement. Note also that for successive measurements it holds the composition property

$$\mathcal{C}^{\epsilon}_{a|A}\mathcal{C}^{\delta}_{a|A} = (1-\epsilon)(1-\delta)\rho + (1-\epsilon)\delta\mathcal{C}_{a|A} + \epsilon(1-\delta)\mathcal{C}_{a|A} + \epsilon\delta\mathcal{C}_{a|A} = \mathcal{C}^{\epsilon+\delta-\epsilon\delta}_{a|A}. \quad (4.10)$$

Also, by noticing that

$$\mathcal{C}^{\epsilon}_{a|A}(\rho) - \rho = \epsilon[\mathcal{C}_{a|A}(\rho) - \rho], \quad (4.11)$$

one shows the relation

$$\mathcal{C}^{\epsilon}_{a|A}(\rho) - \mathcal{C}^{\delta}_{a|A}(\rho) = (\epsilon-\delta)\left[\mathcal{C}_{a|A}(\rho) - \rho\right], \quad (4.12)$$

which provides information about the distance imposed by the application of two measurements of distinct intensities with the same outcome a.

4.1.1 Monitoring

In Sec. 3.5.2 we used the map Φ_A as a model for an unrevealed projective measurement. Now we introduce a model that has the capability of interpolating between weak and projective unrevealed measurements. Let us consider a system $\mathcal{S}$ with a preparation ρ on $\mathcal{H}_{\mathcal{S}} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$. In terms of the eigenbasis $\{a, A_a\}$ of a generic observable $A = \sum_a aA_a$

acting on $\mathcal{H}_\mathcal{A}$, with $A_a A_{a'} = \delta_{aa'} A_a$ and $A_a = |a\rangle\langle a|$, we can write

$$\rho = \sum_{a,a'} \langle a'|\rho|a\rangle \otimes |a'\rangle\langle a| = \sum_{a,a'} p_{aa'} |a'\rangle\langle a| \otimes \rho_{\mathcal{B}|aa'}. \tag{4.13}$$

Now consider a von Neumann pre-measurement induced by the coupling

$$H(t) = \epsilon\, g(t)\, A \otimes \mathbb{1}_\mathcal{B} \otimes P_\mathcal{X}, \tag{4.14}$$

where $\mathcal{X}$ stands for an extra degree of freedom (an ancilla) that will encode the information about A, $P_\mathcal{X}$ is the momentum operator acting on $\mathcal{H}_\mathcal{X}$, and $\int_0^t g(t')dt' = 1$. By the application of the time-evolution operator

$$U(t) = \exp\left[-\tfrac{i}{\hbar}\int_0^t \mathrm{d}t' H(t')\right] \tag{4.15}$$

on the initial state $\rho \otimes |x_0\rangle\langle x_0|$ we get the following joint state in $\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{X}$:

$$\rho_{\mathcal{S}\mathcal{X}}(t) = \sum_{a,a'} p_{aa'} |a'\rangle\langle a| \otimes |x_0 + \epsilon\, a'\rangle\langle x_0 + \epsilon\, a| \otimes \rho_{\mathcal{B}|aa'}. \tag{4.16}$$

Tracing the ancilla gives

$$\rho_\mathcal{S}(t) = \mathrm{Tr}_\mathcal{X}[\rho_{\mathcal{S}\mathcal{X}}(t)] = \sum_{a,a'} \gamma_{aa'}(\epsilon)\, |a'\rangle\langle a| \otimes \rho_{\mathcal{B}|aa'}, \tag{4.17}$$

where $\gamma_{aa'}(\epsilon) = \langle x_0 + \epsilon\, a | x_0 + \epsilon\, a'\rangle$. This term may or may not be small; it depends on the magnitude of the ratio between the distance $\epsilon(a - a')$ and the width of the wave function associated to $|x_0\rangle$. We then consider the model

$$\gamma_{aa'}(\epsilon) = (1-\epsilon) + \epsilon\, \delta_{aa'} \qquad [\forall \epsilon \in (0,1)], \tag{4.18}$$

which continuously connects a scenario of no interaction ($\epsilon \to 0$) with a maximally entangling one ($\epsilon \to 1$). With that, we obtain

$$\rho_\mathcal{S}(t) = (1-\epsilon)\rho + \epsilon\, \Phi_A(\rho). \tag{4.19}$$

This result leads us to introduce the linear map

$$\mathcal{M}_A^\epsilon(\rho) := (1-\epsilon)\,\rho + \epsilon\,\Phi_A(\rho), \tag{4.20}$$

with $\epsilon \in (0,1)$. We refer to $\mathcal{M}_A^\epsilon$ as a *monitoring* with intensity ϵ of A by $\mathcal{X}$. Actually, the relation

$$\mathrm{Tr}_{\mathcal{X}}\rho_{\mathcal{S}\mathcal{X}}(t) = \mathcal{M}_A^\epsilon(\rho) \tag{4.21}$$

is a mere expression of Stinespring's dilation theorem [59]. Notice that

$$\begin{aligned} &\mathcal{M}_A^\epsilon(\rho) = \rho && \text{(no monitoring of } A\text{)},\\ &\mathcal{M}_A^\epsilon(\rho) = \rho - \epsilon[\rho - \Phi_A(\rho)] && \text{(weak monitoring)},\\ &\mathcal{M}_A^\epsilon(\rho) = \Phi_A(\rho) && \text{(strong monitoring)}. \end{aligned} \tag{4.22}$$

As for the weak map with revealed outcomes, we now derive the mathematical properties of $\mathcal{M}_A^\epsilon$. First, we note that because $\mathrm{Tr}\rho = \mathrm{Tr}\Phi_{A_1}(\rho) = 1$, the map $\mathcal{M}_A^\epsilon$ preserves the trace. Another point to note is that the map preserves hermiticity. Because $\Phi_A(\rho)$ is linear, the map $\mathcal{M}_{\epsilon,A_1}$ is also linear, i.e., $\mathcal{M}_A^\epsilon(\rho_1+\rho_2) = \mathcal{M}_A^\epsilon(\rho_1) + \mathcal{M}_A^\epsilon(\rho_2)$. Using these properties, we can study successive applications of the map and use recursion to write:

$$[\mathcal{M}_A^\epsilon]^n(\rho) = \Phi_A(\rho) + (1-\epsilon)^n\big[\rho - \Phi_A(\rho)\big] = (1-\epsilon)^n\rho + \big[1-(1-\epsilon)^n\big]\Phi_A(\rho). \tag{4.23}$$

Notice that $[\mathcal{M}_A^{\epsilon\to 1}]^n(\rho) = \Phi_A(\rho)$, as expected. Also, one has that

$$[\mathcal{M}_A^\epsilon]^n(\rho) = \mathcal{M}_A^{1-(1-\epsilon)^n}(\rho), \tag{4.24}$$

which shows that n monitorings of intensity ϵ is equivalent to a single monitoring of intensity $1-(1-\epsilon)^n$. Finally,

$$\lim_{n\to\infty}[\mathcal{M}_A^\epsilon]^n = \Phi_A, \tag{4.25}$$

which means that infinitely many weak monitorings, executed either sequentially or simultaneously, establish the reality of the monitored observable for any state. From the above relations, further composition properties can be derived:

$$\lim_{n\to\infty}[\mathcal{M}_A^{\epsilon/n}]^n = \mathcal{M}_A^{1-e^{-\epsilon}}, \tag{4.26a}$$

$$\mathcal{M}_A^{\delta}\mathcal{M}_A^{\epsilon} = \mathcal{M}_A^{\delta+\epsilon-\delta\epsilon}. \tag{4.26b}$$

for $\{\epsilon, \delta\} \in (0, 1]$ and $n \geqslant 1 \in \mathbb{Z}$. Also, by noticing that

$$\mathcal{M}_A^{\epsilon}(\rho) - \rho = \epsilon[\Phi_{A_1}(\rho) - \rho], \tag{4.27}$$

one shows that

$$\mathcal{M}_A^{\epsilon}(\rho) - \mathcal{M}_A^{\delta}(\rho) = (\epsilon - \delta)[\Phi_{A_1}(\rho) - \rho], \tag{4.28}$$

which provides information about the distance imposed by the application of two monitorings with distinct intensities. Also, we can write $\mathcal{M}_A^{\epsilon}(\rho) = \rho - \epsilon[\rho - \Phi_A(\rho)]$, which clearly expresses a degradation of the off-diagonal terms of ρ. This is expected since $\mathcal{M}_A^{\epsilon}$ represents a quantum-noise channel. To see this one can set $K_0 = \sqrt{1-\epsilon}\,\mathbb{1}$ and $K_a = \sqrt{\epsilon}\,A_a$ and then write

$$\mathcal{M}_A^{\epsilon}(\rho) = \sum_a K_a \rho K_a^{\dagger} \tag{4.29}$$

with $\sum_a K_a^{\dagger}K_a + K_0^{\dagger}K_0 = \mathbb{1}$, which reveal the operator-sum representation typical of quantum operations [59]. As such, it is clear that $\mathcal{M}_A^{\epsilon}$ is a completely positive trace-preserving (CPTP) map. Since $\Phi_A\Phi_B = \Phi_B\Phi_A$ for arbitrary observables acting on different subspaces, note that it is also true for monitoring

$$\begin{aligned}
\mathcal{M}_A^{\epsilon}\mathcal{M}_B^{\delta} &= \mathcal{M}_A^{\epsilon}\big((1-\delta)\rho + \delta\Phi_B(\rho)\big) \\
&= (1-\epsilon)[(1-\delta)\rho + \delta\Phi_B(\rho)] + \epsilon[(1-\delta)\Phi_A(\rho) + \delta\Phi_A\Phi_B(\rho)] \\
&= (1-\delta)[(1-\epsilon)\rho + \epsilon\Phi_A(\rho)] + \delta\Phi_B[(1-\epsilon)(\rho) + \epsilon\Phi_A] = \mathcal{M}_B^{\delta}\mathcal{M}_A^{\epsilon}
\end{aligned} \tag{4.30}$$

of arbitrary observables A and B. It is also easy to check that:

$$\begin{aligned}
[\mathcal{M}_A^{\epsilon}]^n\big(\Phi_A(\rho)\big) &= (1-\epsilon)^n\Phi_A(\rho) + \big[1 - (1-\epsilon)^n\big]\Phi_A(\rho) \\
&= \Phi_A(\rho) = \Phi_A\big(\mathcal{M}_{\epsilon,A_1}^{n}(\rho)\big).
\end{aligned} \tag{4.31}$$

wich proves the hierarchy of the map Φ_A over $[\mathcal{M}_A^\epsilon]^n$. Interesting this shows that the map $\mathcal{M}_A^\epsilon$ commutes with the map Φ_A for all intensities ϵ! Another point to note is that monitoring never decreases the entropy of the state. We have, through the concavity of von Neumann's entropy and non-negativity of the measure of irreality $\mathfrak{I}(A|\rho)$,

$$S(\mathcal{M}_A^\epsilon(\rho)) \geqslant (1-\epsilon)\, S(\rho) + \epsilon\, S(\Phi_A(\rho)) = S(\rho) + \epsilon\, \mathfrak{I}(A|\rho) \geqslant S(\rho), \tag{4.32}$$

equality for $\epsilon = 0$. By successive applications, it follows that:

$$S([\mathcal{M}_A^\epsilon]^n(\rho)) \geqslant S([\mathcal{M}_A^\epsilon]^{n-1}(\rho)). \tag{4.33}$$

Now we are in position to formally link measurement with monitoring. When an observer knows that a measurement of A of a generic intensity ϵ has been performed on a preparation ρ but is not informed about the outcome a in a given run of the experiment, the only prediction that can be made by this observer is that the state was reduced to some $\mathcal{C}_{a|A}^\epsilon(\rho)$ with probability $p_a = \mathrm{Tr}\,[(A_a \otimes \mathbb{1}_\mathcal{B})\,\rho]$. Without information about the specific outcome a, it follows from the definition (4.2) that the better prediction the observer can make is

$$\begin{aligned}\sum_a p_a \mathcal{C}_{a|A}^\epsilon(\rho) &= (1-\epsilon)\sum_a p_a \rho + \epsilon \sum_a p_a \mathcal{C}_{a|A}(\rho)) \\ &= (1-\epsilon)\rho + \epsilon\Phi_A = \mathcal{M}_A^\epsilon(\rho).\end{aligned} \tag{4.34}$$

Conceptually, there is an important point to make, namely, that monitoring is indistinguishable from an unrevealed measurement. The left-hand side of the above relation was constructed with the basis on a measurement [eventually a collapsing one (for $\epsilon \to 1$)] that has been secretly conducted. The right-hand side, in its turn, was derived via an entangling dynamics with an ancilla, without any *a priori* link with the state reduction. This points out that unrevealed collapse is formally equivalent to entanglement plus discard, which suggests that the state vector reduction can be interpreted as information updating rather than as a physical reduction of the state vector. We will return to this point later within an informational perspective.

An important point to note is that the entropy of the state after monitoring never decreases, which is in agreement with the fact that this model corresponds to a measurement process that is imprecise and, more importantly, the outcomes are never revealed. Next, we intend to discuss the effects and the relation of the physical procedures adopted here with BA's measure of irreality.

We are now ready to present the main contribution of this thesis, namely, the formal development of connections between the notion of reality, measurement, information, and quantum correlations in weak-disturbance scenarios. Through the increment of reality induced by monitoring we will be led to propose a quantifier for weak reality and show how it connects with the strong element of reality defined by BA.

4.2 MEASUREMENT, INFORMATION, AND REALITY

4.2.1 Monitoring increases reality

Consider a preparation ρ on $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$. For this state, the degree of irreality of a generic observable A is given by $\mathfrak{I}(A|\rho)$. Under a monitoring $\mathcal{M}_A^\epsilon$ of arbitrary intensity ϵ, the irreality of A changes to $\mathfrak{I}(A|\mathcal{M}_A^\epsilon(\rho))$. Although a quantifier $\mathfrak{R}$ of reality itself has not been defined so far in the literature, it is clear that this concept should be dual to irreality, that is,

$$\Delta\mathfrak{I}(A) + \Delta\mathfrak{R}(A) = 0. \tag{4.35}$$

Then, under the monitoring $\mathcal{M}_A^\epsilon$ on ρ, the reality of A changes as

$$\Delta\mathfrak{R}(A) := -\Delta\mathfrak{I}(A) = \mathfrak{I}(A|\rho) - \mathfrak{I}(A|\mathcal{M}_A^\epsilon(\rho)). \tag{4.36}$$

It follows that

$$\begin{aligned} \Delta\mathfrak{R}(A) &= S(\Phi_A(\rho)) - S(\rho) - S(\Phi_A(\mathcal{M}_A^\epsilon(\rho))) + S(\mathcal{M}_A^\epsilon(\rho)) && (4.37)\\ &= S(\mathcal{M}_A^\epsilon(\rho)) - S(\rho), && (4.38) \end{aligned}$$

which is a non-negative quantity. If $\rho = \Phi_A(\rho)$, then $\Delta\mathfrak{R} = 0$, since in this case the preparation ρ is already a state of reality for A. If $\epsilon \to 1$, then the reality change saturates to its maximum value $\Delta\mathfrak{R}_{\max}(A) = \mathfrak{I}(A|\rho)$, meaning that the reality increases precisely by the value that defined the amount by which the observable was unreal. From the concavity of the von Neumann entropy we can write

$$S(\mathcal{M}_A^\epsilon(\rho)) \geqslant (1-\epsilon)S(\rho) + \epsilon S(\Phi_A(\rho)) \tag{4.39}$$

and using the non-negativity of irreality we obtain

$$\Delta\mathfrak{R}(A) \geqslant \epsilon\, \mathfrak{I}(A|\rho), \tag{4.40}$$

with the equality holding for $\epsilon \to 1$. [Actually, the equality also holds for $\epsilon \to 0$ and $\rho = \Phi_A(\rho)$, but in these cases both the left-hand-side and right-hand-side terms vanish.] Hence, apart from extremal instances, the reality of an observable typically increases under monitoring.

Furthermore, we show that under monitoring the reality increase is bounded from above. To do so, we invoke the Fannes-Audenaert inequality [115], which is an refinement of the so called Fanne's inequality that gives an upper bound on the absolute value of the difference between the von Neumann entropies of two finite-dimensional quantum states, in terms of their trace norm distance. The Fannes-Audenaert inequality states that

$$|S(\rho) - S(\sigma)| \leqslant T \ln(d-1) + H(T), \tag{4.41}$$

where $T(\rho,\sigma) = \frac{1}{2}\mathrm{Tr}||\rho - \sigma||_1 \in [0,1]$ is the trace norm,

$$H(T) = -T\ln T - (1-T)\ln(1-T) \tag{4.42}$$

is the Shannon entropy, $||\varrho||_1 = (\varrho^\dagger \varrho)^{1/2}$ is the Schatten 1-norm, and $d = \dim \mathcal{H}$. Using the relation (4.28),

$$\mathcal{M}_A^\epsilon(\rho) - \rho = \epsilon(\Phi_A(\rho) - \rho), \tag{4.43}$$

we can write

$$T(\mathcal{M}_A^\epsilon(\rho), \rho) = \epsilon\,\tau, \tag{4.44}$$

with $\tau \equiv T(\Phi_A(\rho), \rho)$. With that, it is easy to see that the following is true:

$$\Delta\mathfrak{R}(A) \leqslant \epsilon\,\tau \ln (d-1) + H\left(\epsilon\,\tau\right). \tag{4.45}$$

It can be checked for $\tau > 0$ that the above upper bound can never reach the value $d\sqrt{\epsilon\,\tau/e}$, which can therefore be taken as a simpler estimate for the $\Delta\mathfrak{R}(A)$ upper bound. The inequalities (4.40) and (4.45) define our first result: A monitoring of intensity ϵ, which can be interpreted either as an unrevealed measurement or as an operation involving entanglement plus discard, implies a finite increase not less than $\epsilon\mathfrak{I}(A|\rho)$ in the reality of the monitored observable. Notice that the increment in the reality, whose upper bound is regulated by the monitoring intensity ϵ, can be made to be infinitesimal.

After measuring σ_z for a spin-$\frac{1}{2}$ particle prepared in a generic state ρ and announcing the result, the state of the system collapses to one of the states $|\pm z\rangle = \left(|+x\rangle \pm |-x\rangle\right)/\sqrt{2}$. Thus, while the reality of σ_z increases in the process, the irreality of an incompatible observable, say, σ_x, reaches its maximum value, so its reality decreases. As we show now, the situation is rather different when a monitoring is involved.

4.2.2 Monitoring and incompatible observables

Let A and A' be incompatible observables acting on $\mathcal{H}_\mathcal{A}$. We want to see how the reality of A' changes when a monitoring $\mathcal{M}_A^\epsilon$ of A is performed on $\rho \in \mathcal{H}_\mathcal{S}$. Via the relations (3.56) and (4.36), the reality change $\Delta\mathfrak{R}(A') = \mathfrak{I}(A'|\rho) - \mathfrak{I}(A'|\mathcal{M}_A^\epsilon(\rho))$ can be written in the form

$$\Delta\mathfrak{R}(A') = S(\Phi_{A'}(\rho)) + S(\mathcal{M}_A^\epsilon(\rho)) - S(\rho) - S(\Phi_{A'}\mathcal{M}_A^\epsilon(\rho)). \tag{4.46}$$

To infer the behavior of this quantity, we consider an extended space $\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{X} \otimes \mathcal{H}_\mathcal{Y}$, with $\mathcal{H}_\mathcal{S} = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$, and write

$$\rho_{\mathcal{S}\mathcal{X}\mathcal{Y}} = U_{\mathcal{S}\mathcal{X}} U_{\mathcal{S}\mathcal{Y}} \Big(\rho \otimes |x_0\rangle\langle x_0| \otimes |y_0\rangle\langle y_0|\Big) U_{\mathcal{S}\mathcal{Y}}^\dagger U_{\mathcal{S}\mathcal{X}}^\dagger, \tag{4.47}$$

with unitary transformations such that

$$U_{\mathcal{S}\mathcal{X}} = e^{-\frac{i\epsilon}{\hbar} A\otimes \mathbb{1}_{\mathcal{B}} \otimes P_{\mathcal{X}}}, \qquad U_{\mathcal{S}\mathcal{Y}} = e^{-\frac{i\delta}{\hbar} A'\otimes \mathbb{1}_{\mathcal{B}} \otimes P_{\mathcal{Y}}}, \tag{4.48}$$

and $[U_{\mathcal{S}\mathcal{X}}, U_{\mathcal{S}\mathcal{Y}}] \neq 0$. These operators refer to von Neumann pre-measurements of the observables A and A', with intensities ϵ and δ, via ancillary systems $\mathcal{X}$ and $\mathcal{Y}$, respectively. From the relations above and the Stinespring dilation theorem [see also Eq. (4.20)] one may directly obtain the reduced state

$$\rho_{\mathcal{S}\mathcal{X}} = U_{\mathcal{S}\mathcal{X}} \left(\mathcal{M}^{\delta}_{A'}(\rho) \otimes |x_0\rangle\langle x_0|\right) U^{\dagger}_{\mathcal{S}\mathcal{X}}, \tag{4.49}$$

which by unitary invariance of the von Neumann entropy implies that $S(\rho_{\mathcal{S}\mathcal{X}}) = S(\mathcal{M}^{\delta}_{A'}(\rho))$. For the same reason, $S(\rho_{\mathcal{S}\mathcal{X}\mathcal{Y}}) = S(\rho)$. To compute the reduction $\rho_{\mathcal{S}\mathcal{Y}}$ we first note that

$$U_{\mathcal{S}\mathcal{X}}U_{\mathcal{S}\mathcal{Y}} = U_{\mathcal{S}\mathcal{X}} \left(e^{-\frac{i\delta}{\hbar} A'\otimes \mathbb{1}_{\mathcal{B}} \otimes P_{\mathcal{Y}}}\right) U^{\dagger}_{\mathcal{S}\mathcal{X}} U_{\mathcal{S}\mathcal{X}} = e^{-\frac{i\delta}{\hbar} \tilde{A}'\otimes \mathbb{1}_{\mathcal{B}} \otimes P_{\mathcal{Y}}} U_{\mathcal{S}\mathcal{X}},$$

where $\tilde{A}' \otimes \mathbb{1}_{\mathcal{B}} = U_{\mathcal{S}\mathcal{X}}(A' \otimes \mathbb{1}_{\mathcal{B}})U^{\dagger}_{\mathcal{S}\mathcal{X}}$. Because $\tilde{A}'$ is Hermitian, we have thus shown that $U_{\mathcal{S}\mathcal{X}}U_{\mathcal{S}\mathcal{Y}} = \tilde{U}_{\mathcal{S}\mathcal{Y}}U_{\mathcal{S}\mathcal{X}}$, with a new unitary operator $\tilde{U}_{\mathcal{S}\mathcal{Y}}$. With this result, we can turn to Eq. (4.47) to show that

$$\rho_{\mathcal{S}\mathcal{Y}} = \tilde{U}_{\mathcal{S}\mathcal{Y}} \left(\mathcal{M}^{\epsilon}_{A}(\rho) \otimes |y_0\rangle\langle y_0|\right) \tilde{U}^{\dagger}_{\mathcal{S}\mathcal{Y}}. \tag{4.50}$$

It thus follows that $S(\rho_{\mathcal{S}\mathcal{Y}}) = S(\mathcal{M}^{\epsilon}_{A}(\rho))$. Since $\Phi_A\Phi_{A'} = \Phi_{A'}\Phi_A = \frac{\mathbb{1}}{d}$ it can be observed that

$$\begin{aligned}
\mathcal{M}^{\epsilon}_{A}\mathcal{M}^{\delta}_{A'} &= \mathcal{M}^{\epsilon}_{A}\big((1-\delta)\rho + \delta\Phi_{A'}(\rho)\big) \qquad (4.51)\\
&= (1-\epsilon)[(1-\delta)\rho + \delta\Phi_{A'}(\rho)] + \epsilon[(1-\delta)\Phi_A(\rho) + \delta\Phi_A\Phi_{A'}(\rho)]\\
&= (1-\delta)[(1-\epsilon)\rho + \epsilon\Phi_A(\rho)] + \delta\Phi_{A'}[(1-\epsilon)(\rho) + \epsilon\Phi_A(\rho)] = \mathcal{M}^{\delta}_{A'}\mathcal{M}^{\epsilon}_{A} = \rho_{\mathcal{S}}
\end{aligned}$$

Then, from the strong subadditivity of the von Neumann entropy

$$S(\rho_{\mathcal{S}\mathcal{X}\mathcal{Y}}) + S(\rho_{\mathcal{S}}) \leqslant S(\rho_{\mathcal{S}\mathcal{X}}) + S(\rho_{\mathcal{S}\mathcal{Y}}) \tag{4.52}$$

we arrive at

$$S(\rho) + S(\mathcal{M}^{\delta}_{A'}\mathcal{M}^{\epsilon}_{A}(\rho)) \leqslant S(\mathcal{M}^{\delta}_{A'}(\rho)) + S(\mathcal{M}^{\epsilon}_{A}(\rho)). \tag{4.53}$$

Given that $\mathcal{M}^{\delta\to 1}_{A'} = \Phi_{A'}$, we return to Eq. (4.46) to obtain

$$\Delta\mathfrak{R}(A') \geqslant 0, \tag{4.54}$$

with equality holding for $\{\epsilon, \delta\} \to 0, 1$ and $\rho = \Phi_{A(A')}(\rho)$. This result is surprising, as it shows that under monitoring of A the reality of A' will also increase in general. In fact, along with the inequality (4.40), this shows that a monitoring typically increases the global reality of a system.

It is worth mentioning that the inequalities (4.40) and (4.54), along with some results reported in Ref. [53], prove the monotonicity of BA's irreality under monitoring (a CPTP map), that is, $\mathfrak{I}(A|\rho) \geqslant \mathfrak{I}(A|\mathcal{M}^{\epsilon}_{O}(\rho))$ for $\rho \in \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$ and O being a generic Hermitian operator acting on $\mathcal{H}_{\mathcal{A}}$ or $\mathcal{H}_{\mathcal{B}}$ and A an Hermitian operator acting on $\mathcal{H}_{\mathcal{A}}$. This means that the irreality never increases under monitoring. This observation naturally raises the following question: Is there any scenario in which the irreality of an observable can increase? Next we address this question.

4.3 REVEALED MEASUREMENTS AND IRREALITY

Consider two maximally incompatible observables A and A' acting on $\mathcal{H}_{\mathcal{A}}$, meaning that their eigenstates constitute mutually unbiased bases satisfying $|\langle a|a'\rangle|^2 = 1/d_{\mathcal{A}}$, where $d_{\mathcal{A}} = \dim \mathcal{H}_{\mathcal{A}}$. Let $\rho_{[A']}$ denote a reality state for A', that is, $\rho_{[A']} = \Phi_{A'}(\rho)$ and therefore $\mathfrak{I}(A'|\rho_{[A']}) = 0$. Under monitoring of the incompatible observable A the state transforms to

$$\mathcal{M}^{\epsilon}_{A}(\rho_{[A']}) = (1-\epsilon)\Phi_{A'}(\rho) + \tfrac{\mathbb{1}}{d_{\mathcal{A}}} \otimes \rho_{\mathcal{B}}, \tag{4.55}$$

where $\rho_{\mathcal{B}} = \mathrm{Tr}_{\mathcal{A}}(\rho)$. Since this state does not change under $\Phi_{A'}$ we can check that $\mathfrak{I}(A'|\mathcal{M}^{\epsilon}_{A}(\rho_{[A']})) = 0$. This result shows that the monitoring of A does not increase the irreality of the incompatible observable A'. As such, it is an illustration of the more general result obtained in the preceding section. Interestingly, now we show that this situation changes when weak revealed measurements are involved. To this end, let us invoke the map $\mathcal{C}^{\epsilon}_{a|A}$, which was introduced in Eq. (4.2) as an effective descriptor for a measurement of A with generic intensity ϵ and known outcome a. In this case, we have

$$\mathcal{C}^{\epsilon}_{a|A}(\rho_{[A']}) = (1-\epsilon)\Phi_{A'}(\rho) + \epsilon A_a \otimes \rho_{\mathcal{B}}, \tag{4.56}$$

which *does* change under the map $\Phi_{A'}$ since $\Phi_{A'}(A_a) = \frac{\mathbb{1}}{d_{\mathcal{A}}}$. It then follows that

$$\mathfrak{I}(A'|\mathcal{C}^{\epsilon}_{a|A}\rho_{[A']}) > 0, \tag{4.57}$$

that is, the irreality of A' indeed increases under revealed measurements. In addition, by direct application of Fannes's inequality [see the inequality (4.45) and its derivation] one may show that

$$\mathfrak{I}(A'|\mathcal{C}^{\epsilon}_{a|A}\rho_{[A']}) \leqslant \epsilon\,\tilde{\tau}\ln(d-1) + H(\epsilon\,\tilde{\tau}), \tag{4.58}$$

where $\tilde{\tau} \equiv T\left(\frac{\mathbb{1}}{d_{\mathcal{A}}}, A_a\right) = 1 - 1/d_{\mathcal{A}}$. Again we can take $d\sqrt{\epsilon\,\tilde{\tau}/e}$ as a simpler estimate for the upper bound given above. We have thus proved that irreality can be generated for A' by means of revealed measurements of the incompatible observable A. Being controlled by the measurement intensity ϵ, it is clear that the generated irreality can be made arbitrarily small.

Given the generality of the previous results, that is, monitoring does not decrease the reality of both the observables being measured and of an incompatible and that measures with revealed results increase the irreality of the observable incompatible, we will devote the next section to a case study involving revealed measurement and subsequent mixing of results. We will now focus on strong measurements analyzed in an ensemble of quantum states, which will be important for the end of the thesis to discuss Everett's paradox.

4.4 REALITY AND ERASE OF INFORMATION

Consider an ensemble E_0 with a number N of copies of a given physical system (the system may contain several particles). Suppose that measurements of A_1 on the particle 1 of each copy are made, then producing the state

$$\mathcal{C}_{k|A_1}(\rho) = \frac{A_{1k}\rho A_{1k}}{p_k} = A_{1k} \otimes \rho_{\mathcal{B}|k} \ . \tag{4.59}$$

If there are N_k occurrences of the eigenvalue a_{1k}, we can form a subensemble E_k and then estimate the probability $p_k = \frac{N_k}{N}$ of occurrence of this eigenvalue. The ignorance associated with the state pertinent to the sub-ensemble E_k is $S(\mathcal{C}_{k|A_1}(\rho))$. The average ignorance per system, accumulated after such measurements A_1 over E_0, is given by

$$\bar{\mathcal{S}}(E_1, E_2, ...) \equiv \sum_k p_k S(\mathcal{C}_{k|A_1}(\rho)) \tag{4.60}$$

$$= \sum_k p_k S(A_{1k} \otimes \rho_{\mathcal{B}|k}) = \sum_k p_k S(\rho_{\mathcal{B}|k}). \tag{4.61}$$

Now suppose the ensembles E_k, each one with a known state $\mathcal{C}_{k|A_1}(\rho)$, are mixed, forming a resultant ensemble $E_1 + E_2 + ...$, which shall have the same number of copies of E_0. A system from this ensemble will be

$$\Phi_{A_1}(\rho) = \sum_k p_k \mathcal{C}_{k|A_1}(\rho), \tag{4.62}$$

while for the inicial ensemble E_0 we have ρ. For any copy in the resulting ensemble, the associated entropy will be

$$\mathcal{S}(E_1 + E_2 + ...) = S(\sum_k p_k \mathcal{C}_{k|A_1}(\rho)) \tag{4.63}$$

$$= S(\sum_k p_k A_{1k} \otimes \rho_{\mathcal{B}|k}) = H(\{p_k\}) + \sum_k p_k S(\rho_{\mathcal{B}|k}),$$

where $H(\{p_k\})$ is the Shannon entropy of the distribution of subensembles $\{p_k\}$. Therefore, the process of mixing (information erasure)increases the ignorance by a value

$$\begin{aligned}\Delta\mathcal{S} &\equiv \mathcal{S} - \bar{\mathcal{S}} \\ &= S(\sum_k p_k \mathcal{C}_{k|A_1}(\rho)) - \sum_k p_k S(\mathcal{C}_{k|A_1}(\rho)) \\ &= S(\Phi_{A_1}(\rho_{\mathcal{A}})) = H(\{p_k\}).\end{aligned} \tag{4.64}$$

This result is equivalent to the ignorance $S(\Phi_{A_1}(\rho_{\mathcal{A}}))$ associated with unread measurements on the subsystem A. Clearly, the omission of individual results implies an increase in the ignorance about the ensemble. For the average irreality, one has

$$\bar{\mathfrak{I}}(A_1|E_1\ , E_2\ ...) \equiv \sum_k p_k \mathfrak{I}(A_1|\mathcal{C}_{k|A_1}(\rho) = 0, \tag{4.65}$$

as expected. As nothing is said about the original ensemble E_0, for which the state of each particle is ρ, we can immediately conclude that

$$\bar{\mathfrak{I}}(A_1|E_1, E_2...) \leq \mathfrak{I}(A_1|E_0), \tag{4.66}$$

i.e., the measurement increases the reality of A_1, as expected. On the other hand, given A_2 maximally incompatible with A_1, we will have

$$\bar{\mathfrak{I}}(A_2|E_1, E_2...) \equiv \sum_k p_k \mathfrak{I}(A_2|\mathcal{C}_{k|A_1}(\rho)) = \ln d_{\mathcal{A}}, \tag{4.67}$$

since that $\mathfrak{I}(A_2|\mathcal{C}_{k|A_1}(\rho)) = \ln d_{\mathcal{A}}$.

Also, since

$$\mathfrak{I}(A_2|\rho_{\mathcal{A}}) \leq \ln d_{\mathcal{A}}, \tag{4.68}$$

and

$$\mathfrak{I}(A|\rho) = \mathfrak{I}(A|\rho_{\mathcal{A}}) + D_A(\rho) \tag{4.69}$$

we have for the cases in which there are no quantum correlations between the parties, this is $D_A(\rho) = 0$, we can write

$$\mathfrak{I}(A_2|\rho) \leq \ln d_{\mathcal{A}}. \tag{4.70}$$

For this particular case, we can return to write,

$$\bar{\mathfrak{I}}(A_2|E_1\ , E_2\ ...) \geq \mathfrak{I}(A_2|E_0). \tag{4.71}$$

So we see that if we do not have quantum correlations between the parties, the measurement of A_1 in an ensemble is a way of increasing the irreality of A_2.

Let us now look at the effect of mixing subensembles if we consider that we start with a state $\sigma = \Phi_{A_2}(\rho)$. Composing $E = E_1 + E_2 + ...$, we obtain $\Phi_{A_1} = \sum_k p_k \mathcal{C}_{k|A_1}(\sigma)$. The irreality A_1 per system will be

$$\mathfrak{I}(A_1|E_1 + E_2 +\ \ ...) \equiv \mathfrak{I}(A_1|\sum_k p_k \mathcal{C}_{k|A_1}(\sigma)) = 0. \tag{4.72}$$

Therefore, the deletion of information does not cause any change in the irreality in A_1, after all, after the projection $\mathcal{C}_{k|A_1}(\sigma)$), the reality of A_1 was already established. About the irreality of A_2 after the mixture, one can show, using the joint entropy theorem [59], the definition of mutual information and $H(\{p_k\}) = S(\Phi_{A_1}(\rho_{\mathcal{A}}))$ that

$$\begin{aligned} \mathfrak{I}(A_2|E_1 + E_2 + ...) &\equiv \mathfrak{I}(A_2|\sum_k p_k \mathcal{C}_{k|A_1}(\sigma) \\ &= \ln d_{\mathcal{A}} - H(\{p_k\}) + S(\sigma_{\mathcal{B}}) - \sum_k p_k S(\sigma_{\mathcal{B}|k}) \\ &= \ln d_{\mathcal{A}} - S(\Phi_{A_1}(\sigma_{\mathcal{A}})) + I_{A:B}(\Phi_{A_1}(\sigma)). \end{aligned} \tag{4.73}$$

This result provides

$$\begin{aligned} \Delta\bar{\mathfrak{I}}_{A_2} &= \mathfrak{I}(A_2|E_1 + E_2+, ...) - \bar{\mathfrak{I}}(A_2|E_1, E_2, ...) \\ &= I_{A:B}(\Phi_{A_1}(\sigma)) - \Delta\mathcal{S} \leq 0. \end{aligned} \tag{4.74}$$

In this case, the variation of the irreality of A_2 is numerically equal to the mutual information that B maintains with A after the measurement of A_1 followed by the mixture, subtracted from the amount of information of A_1 which is lost by the realization of such a mixture. Note that, in this case, the reality of both A_1 and A_2 never decreases with the mixture of subensembles.

So far, we have explored the generation of irreality through revealed or non-revealed measurement processes, but is it possible to dynamically generate irreality in nature via unitary interactions (note that measurements involve non-unitary, such as partial traces)? The answer to this is based on a frustrated conservation law, which will be explained in following.

4.5 FRUSTRATION OF CONSERVATIONS LAWS

We now assess the possibility of generating irreality in unitary dynamics. Consider a preparation ρ on $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$. Let $U_{\mathcal{B}}$ be a unitary transformation acting on $\mathcal{H}_{\mathcal{B}}$. Since Φ_A commutes with $U_{\mathcal{B}}$ it follows that $\mathfrak{I}(A|U_{\mathcal{B}}\rho U_{\mathcal{B}}^{\dagger}) = \mathfrak{I}(A|\rho)$. This shows that a local unitary transformation is not able to promote an increase of irreality in a remote site. We are left then with global unitary transformations.

In what follows we will conduct our analysis in terms of a concrete example involving the frontal scattering of a particle of mass m, initially prepared in a Gaussian wave packet of mean momentum $p_0 = mv_0$ and width $\Delta p = m\Delta v$, by a molecule of mass M, prepared in a Gaussian wave packet of null mean momentum and width $\Delta P = M\Delta v$. Assuming that the probability of the scattering to occur is $1/2$ we have the following state after the interaction

$$|p_0\rangle|0\rangle \to \frac{1}{\sqrt{2}}(|p_0\rangle|0\rangle + |p\rangle|P\rangle), \tag{4.75}$$

the notation is such that "$|p\rangle|P\rangle$" represents a product of wave packets with mean momentum p and variance $(\Delta p)^2$ for the particle and P and $(\Delta P)^2$ for the molecule, respectively. Considering that the collision is elastic, then the nonrelativistic energy and momentum conservation laws lead the following constraints,

$$mv_0 = MV - mv \tag{4.76}$$

and

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{1}{2}MV^2. \tag{4.77}$$

Defining $\xi = m/M$, we can easily write the following expressions for the final velocity v of the particle with mass m

$$v = \frac{1-\xi}{1+\xi}v_0 \tag{4.78}$$

and V for the molecule with mass M

$$V = \xi(\frac{2}{1+\xi})v_0. \tag{4.79}$$

So, up to a normalization factor, we have the following state after the interaction

$$|p_0\rangle|0\rangle \to |p_0\rangle|0\rangle + |(1-\alpha)p_0\rangle|\alpha p_0\rangle, \tag{4.80}$$

where $\alpha = 2/(1+\xi)$. Via direct calculations one computes the overlaps:

$$O_{\text{part}} \equiv \left|\langle p_0|(1-\alpha)p_0\rangle\right| = \exp\left[-\tfrac{1}{2}\left(\frac{1}{1+\xi}\frac{v_0}{\Delta v}\right)^2\right], \tag{4.81a}$$

$$O_{\text{mol}} \equiv \left|\langle 0|\alpha p_0\rangle\right| = \exp\left[-\tfrac{1}{2}\left(\frac{\xi}{1+\xi}\frac{v_0}{\Delta v}\right)^2\right]. \tag{4.81b}$$

Now, since a measure of irreality for continuous variables is not yet available in the literature, here we approximately treat position and momentum as discrete variables relative to some (experimental) resolutions δ_x and δ_p and then apply the present formalism. Within this framework, if $\Delta p < \delta_p$, then the initial momentum of the particle is effectively real. Let us also assume that $\Delta v \ll v_0$ and consider two regimes. First, if the molecule is not so heavy, so that $\xi \approx 1$, then $O_{\text{part}} \approx O_{\text{mol}} \approx 0$ and the state (4.80) is highly entangled. The relation (3.59) implies, as a consequence of the quantum correlations generated by the scattering, that the irreality of the momentum of the particle has increased. This shows that an entangling unitary dynamics is an effective mechanism to create irreality. On the other hand, if we restrict ourselves to the subsystem particle and thus trace out the molecule degree of freedom, then the resulting reduced state will be the mixture

$|p_0\rangle\langle p_0| + |0\rangle\langle 0|$, which means no irreality whatsoever. Hence, as far as the particle is considered as an individual, there is no increase in the irreality of its momentum. We see here with this example, a clear manifestation of contextuality in QM, that is, the irreality of the system depends on how we treat it, individually or being part of a larger system (particle + molecule). We then move to the second regime of interest. Consider now a very heavy molecule, so that $\xi \to 0$. In this case, $O_{\text{part}} \approx 0$, $O_{\text{mol}} \to 1$, and therefore $|0\rangle \approx |2\alpha p_0\rangle$, meaning that the state of system evolves from $|p_0\rangle|0\rangle$ to $(|p_0\rangle + |-p_0\rangle)|0\rangle$. In other words, while no entanglement is produced between the subsystems, a significant quantum superposition is created. In this case, the local irreality noticeably increases. Notice that because $|0\rangle \approx |2\alpha p_0\rangle$ the time evolution of the global state is such that the momentum conservation seems to have been effectively frustrated.

A similar situation can be formulated in terms of a photon incident on a beam-splitter (BS). If BS is light to the point of moving by interaction with the photon, then passing through BS, the photon will become entangled with BS, by conservation of linear momentum and energy. Even the wavelength of the photon should change according to the Compton effect. If we look only at the momentum of the photon, we will say that it is real, because its reduced state is mixed. In addition, if we use a much heavier BS, then it will no longer store path information on the photon, which will be in a state of superposition of two linear moments (unreal moment).

This mechanism also appears in paradigmatic experiments where local irreality (coherence) is generated. When a particle initially moving with a well-defined momentum $p\,\hat{\imath}$ diffracts through an orifice (a tiny circular slit) it ends up in a superposition of momentum states associated with directions orthogonal to $\hat{\imath}$. In this case, since we cannot detect any motion of the orifice, which is rigidly attached to the laboratory (the reference frame), we have an effective frustration of the momentum conservation law. The situation is similar when the spin of a particle is flipped by a magnet which, being fixed in the laboratory, cannot rotate relatively to this reference frame. Then the observer perceives an effective violation of the total angular momentum conservation. These examples suggest that the frustration of a conservation law within a unitary dynamics is *the* crucial mechanism for the generation of local irreality in interacting dynamics.

4.6 INFORMATION-REALITY COMPLEMENTARITY

A particularly interesting aspect that emerges in the present framework is a clear link between information and reality. Consider an instance in which a system $\mathcal{S}$ initially prepared in a state $\rho_\mathcal{S}$ on $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$ ends up in $\mathcal{M}_A^\epsilon(\rho_\mathcal{S})$ after the monitoring of a generic observable A on $\mathcal{H}_\mathcal{A}$. As mentioned above, the Stinespring theorem ensures that this mapping can be cast in terms of an entangling dynamics $U(t)$ between $\mathcal{S}$ and some extra degree of freedom $\mathcal{X}$ initially prepared in a state $|x_0\rangle\langle x_0|$, that is,

$$\mathcal{M}_A^\epsilon(\rho_\mathcal{S}) = \mathrm{Tr}_\mathcal{X}\left[U(t)\,\rho_\mathcal{S} \otimes |x_0\rangle\langle x_0|\,U^\dagger(t)\right] = \rho_\mathcal{S}(t). \tag{4.82}$$

The mutual information of the joint system $\mathcal{S}\mathcal{X}$ at an arbitrary instant t reads

$$I_{\mathcal{S}:\mathcal{X}}(t) = S(\rho_\mathcal{S}(t)) + S(\rho_\mathcal{X}(t)) - S(\rho_{\mathcal{S}\mathcal{X}}(t)), \tag{4.83}$$

since the joint evolution is unitary, then $S(\rho_{\mathcal{S}\mathcal{X}}(t)) = S(\rho_{\mathcal{S}\mathcal{X}}(0))$. Introducing

$$\Delta S_{\mathcal{S}(\mathcal{X})} = S(\rho_{\mathcal{S}(\mathcal{X})}(t)) - S(\rho_{\mathcal{S}(\mathcal{X})}(0)), \tag{4.84}$$

the change of the mutual information with time reads

$$\Delta I_{\mathcal{S}:\mathcal{X}} = \Delta S_\mathcal{S} + \Delta S_\mathcal{X}. \tag{4.85}$$

Via $I_{\mathcal{S}(\mathcal{X})} = \ln d_{\mathcal{S}(\mathcal{X})} - S(\rho_{\mathcal{S}(\mathcal{X})})$ and Eq. (4.82) we respectively have $\Delta S_\mathcal{X} = -\Delta I_\mathcal{X}$ and $\Delta S_\mathcal{S} = S(\mathcal{M}_A^\epsilon(\rho_\mathcal{S})) - S(\rho_\mathcal{S})$, so that

$$\Delta I_{\mathcal{S}:\mathcal{X}} + \Delta I_\mathcal{X} = S(\mathcal{M}_A^\epsilon(\rho_\mathcal{S})) - S(\rho_\mathcal{S}). \tag{4.86}$$

Using Eq. (4.36) we then arrive at

$$\Delta\left(I_{\mathcal{S}:\mathcal{X}} + I_\mathcal{X}\right) + \Delta\mathfrak{I}(A) = 0. \tag{4.87}$$

From the identity (3.46) and the unitarity of the joint dynamics it follows that $\Delta\left(I_{\mathcal{S}:\mathcal{X}} + I_{\mathcal{X}}\right) = -\Delta I_{\mathcal{S}}$, which allows us to write

$$\Delta I_{\mathcal{S}} + \Delta\mathfrak{R}(A) = 0. \tag{4.88}$$

The relations (4.87) and (4.88) formally state the complementarity between (ir)reality and information, which is another important contribution of this work. As is schematically illustrated in Fig. 4.1, variations in both the local information $I_{\mathcal{X}}$ associated with the subsystem $\mathcal{X}$ and the information $I_{S:X}$ shared by $\mathcal{S}$ and $\mathcal{X}$ directly imply variations in A's irreality. In particular, it is interesting to note that if $\rho_{\mathcal{S}}$ is a pure state, then the joint initial state is pure as well and the entanglement E in the system $\mathcal{S}\mathcal{X}$ is given by $E = S(\rho_{\mathcal{S}(X)}(t))$. Since $\Delta I_{\mathcal{S}} = I_{\mathcal{S}}(t) - I_{\mathcal{S}}(0) = -E$, it follows that

$$\Delta\mathfrak{R}(A) = E, \tag{4.89}$$

which explicitly shows that the reality change in A is determined by the amount of entanglement between $\mathcal{S}$ and $\mathcal{X}$. In other words, because $\mathcal{X}$ gets information about A, this observable becomes real. This is in full agreement with the results reported in Ref. [54], where entanglement is shown to prevent the wavelike behavior of a quantum system.

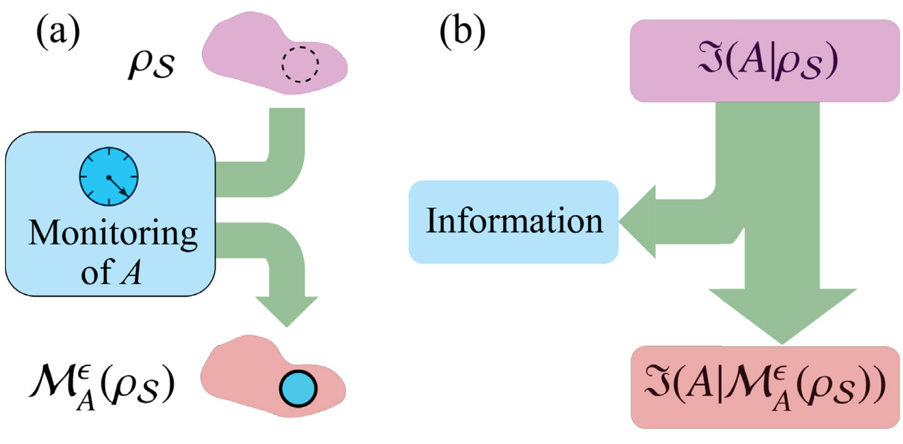

Figure 4.1: (a) Generic state $\rho_{\mathcal{S}}$ on $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$ becomes $\mathcal{M}_A^\epsilon(\rho_{\mathcal{S}})$ under monitoring of an observable A acting on $\mathcal{H}_{\mathcal{A}}$. (b) Same process abstractly pictured in terms of irreality and information. As both local and global information is generated, the irreality of A decreases [see Eq. (4.87)].

In what follows, we discuss a recent experimental implementation of the information-reality complementarity developed in this thesis.

4.6.1 Experimental implementation

Recently, Mancino *et.al* [35] explored the implications of monitoring for the variation of realistic properties of two-level quantum systems in an experiment based on a photonic weak measurement device [35, 111, 112]. In their experiment, they started with a fiducial bipartite state prepared closely to a pure state and assumed that the initial entropies are zero with an error comparable to the experimental uncertainties [35]. Suppose that we want to weakly measure the polarization in the z-direction of a photon in a superposed state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. This can be simulate by using an initial ancilla $|\phi\rangle = (\cos\theta|0\rangle + \sin\theta|1\rangle)$. A weak measurement in the system $|\psi\rangle$ can be done in an interferometric setup that implements a controlled-phase interaction $U = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes \sigma_Z$ plus a strong measurement in the x-direction of the ancilla polarization $|+\rangle, |-\rangle$. The evolved state $|\Psi\rangle$ after the interaction is

$$|\Psi\rangle = U(|+\rangle \otimes |\phi\rangle) = \frac{1}{\sqrt{2}}[|0\rangle(\cos\theta|0\rangle + \sin\theta|1\rangle) + |1\rangle(\cos\theta|0\rangle - \sin\theta|1\rangle)]. \tag{4.90}$$

Note that, for $\theta = 0$ we have the following result:

$$U(|+\rangle \otimes |\phi\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle, \tag{4.91}$$

which means no alteration on the initial state of the system. For $\theta = \frac{\pi}{4}$, we have the following correlated state

$$U(|+\rangle \otimes |\phi\rangle) = \frac{1}{\sqrt{2}}(|0\rangle|+\rangle + |1\rangle|-\rangle), \tag{4.92}$$

now, a strong measurement on the second subsystem gives us full information about the state of the system we want to measure. For $0 \leq \theta \leq \frac{\pi}{4}$ we have that

$$\begin{aligned} |\Psi\rangle\langle\Psi| =& \frac{1}{2}|0\rangle\langle 0|(\cos\theta|0\rangle + \sin\theta|1\rangle)(\cos\theta\langle 0| + \sin\theta\langle 1|) \\ &+ \frac{1}{2}|1\rangle\langle 1|(\cos\theta|0\rangle - \sin\theta|1\rangle)(\cos\theta\langle 0| - \sin\theta\langle 1|) \\ &+ \frac{1}{2}|0\rangle\langle 1|(\cos\theta|0\rangle + \sin\theta|1\rangle)(\cos\theta\langle 0| - \sin\theta\langle 1|) \\ &+ \frac{1}{2}|1\rangle\langle 0|(\cos\theta|0\rangle - \sin\theta|1\rangle)(\cos\theta\langle 0| + \sin\theta\langle 1|). \end{aligned} \tag{4.93}$$

Now, it becomes clear that if we take a partial trace on the ancilla system to look at the reduced state $\rho_{\mathcal{S}}(t) = Tr_{\mathcal{X}}[|\Psi\rangle\langle\Psi|]$, we get,

$$\begin{aligned} \rho_{\mathcal{S}}(t) =& \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) + \frac{\cos 2\theta}{2}(|0\rangle\langle 1| + |1\rangle\langle 0|) \\ =& \rho_{\mathcal{S}}(0) + \frac{\cos 2\theta - 1}{2}(|0\rangle\langle 1| + |1\rangle\langle 0|) \\ =& \cos 2\theta(\rho_{\mathcal{S}}(0)) + (1 - \cos 2\theta)(\Phi_{\sigma_z}(\rho_{\mathcal{S}}(0))) \end{aligned} \tag{4.94}$$

where $\rho_{\mathcal{S}}(0) = |+\rangle\langle +|$, and $\Phi_{\sigma_z}(\rho_{\mathcal{S}}(0)) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$. Making $\epsilon = 1 - \cos 2\theta$, we can finally write

$$\rho_{\mathcal{S}}(t) = (1-\epsilon)\rho_{\mathcal{S}}(0) + \epsilon\Phi_{\sigma_z}(\rho_{\mathcal{S}}(0)) = \mathcal{M}^{\epsilon}_{\sigma_z}(\rho_S(0)). \tag{4.95}$$

This is a realistic example of how to produce a weak monitoring of the observable σ_z in the state $\rho_{\mathcal{S}}(0)$.

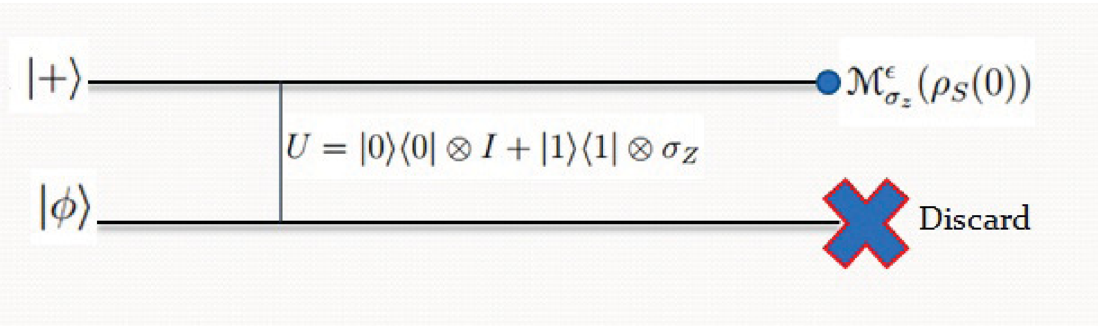

Figure 4.2: A pure state $\rho_{\mathcal{S}}(0) = |+\rangle\langle +|$ becomes $\mathcal{M}^{\epsilon}_{\sigma_z}(\rho_{\mathcal{S}}(0))$ under the action of U followed by discard.

Mancino *et.al* performed full tomography on the final bipartite state $\mathcal{M}^{\epsilon}_{\sigma_z}(\rho_S(0))$ to obtain the relevant quantities in the inequalities developed here. For the variation of the degree of reality induced by monitoring,

$$\Delta\mathfrak{R}(\sigma_z) = S(\mathcal{M}^{\epsilon}_{\sigma_z}(\rho_{\mathcal{S}}(0))) - S(\rho_{\mathcal{S}}(0)), \tag{4.96}$$

the experiment showed that the data follows the predicted behavior for the measured change in the degree of reality of σ_z and clearly satisfies the linear bound

$$\Delta\mathfrak{R}(\sigma_z) \geqslant \epsilon\,\mathfrak{I}(\sigma_z|\rho_{\mathcal{S}}). \tag{4.97}$$

They also evaluated the difference in the information $\Delta I_{\mathcal{S}}$ in order to analyze the complementarity relation $\Delta I_{\mathcal{S}} + \Delta\mathfrak{R}(\sigma_z) = 0$. They argued that the complementarity relation rigorously holds only when the coupling is unitary, hence reversible. This is expected since to deduce our complementarity relation we use the invariance of von Neumann entropy under unitary evolutions, that is:

$$S(\rho_{\mathcal{S}\mathcal{X}}(t)) = S(\rho_{\mathcal{S}\mathcal{X}}(0)). \tag{4.98}$$

In the limit of unitary evolutions, the changes in the degree of reality associated to the observable are the only source of the variations in the mutual information, and in the marginal information content of the ancilla. According to Mancino *et.al*, the experiment shows that the complementarity relation is sensitive to external factors. This comes from the fact that the coupling between the system and the ancilla photons is not unitary, and the dissipation increases with the measurement strength [35]. For more details, we refer the reader to the reference [35].

4.7 REALITY AND QUANTUM CORRELATIONS

After studying how weak monitoring induces weak reality and draw a direct link between the concepts of information and reality, expressed by their complementary character, we have seen that for the case of pure states, the entanglement generated in the interaction is in direct relation to the quantity of reality that has increased after the process. Now, let us take a closer look at the relation between reality and more general quantum correlation such as quantum discord. In this formulation,

$$\mathfrak{I}(A|\rho) = \mathfrak{I}(A|\rho_{\mathcal{A}}) + D_A(\rho), \tag{4.99}$$

which makes clear that the irreality of A is the sum of the local irreality (that is, the irreality of A given the reduced state $\rho_{\mathcal{A}}$) with quantum correlations associated with measurements of A. We may ask ourselves whether the decomposition 4.99, which has been grounded on the projective-measurement map Φ_A, can be generalized in terms of the weak-measurement map $\mathcal{M}_A^\epsilon$. What is the role of arbitrary intensity measurements to define quantum correlations such as quantum discord?

The quantum measurement theory teaches us that when performing a von Neumann measurement, the systems involved get strongly entangled, which produces an one-to-one correlation between the degree of freedom we want to measure and the state of the apparatus. On the other hand, the theory of weak measurements prescribes a measurement interaction that is controlled in such a way that the state of the quantum system of interest is not changed significantly during the interaction. The weak measurements are based on the weak entanglement between the system and the apparatus state, which produces an weak correlation that is not sufficient to induce a collapse in the state that is being measured. Singh and Pati [89] concluded that weak measurements performed on one of the subsystems of a bipartite state can lead to a super quantum discord (SQD) that is always larger than the normal quantum discord captured by the strong measurement. Moreover, they showed that SQD is a monotonic function of measurement strength and it covers all the value between the mutual information and the normal discord. One may argue that this result would be natural if we think that the chosen POVM in the case of weak measurements is not the one that maximizes the classically accessible correlations. However, this is not the case if we consider a distance-based formulation as we shall discuss in the next chapter.

In what follows, we show that by a suitable rewriting of quantum discord in terms of differences between two forms of mutual information it is possible to derive a quantifier of quantum correlations that is weaker than quantum discord when weak measurements are replaced by strong measurements. This definition also agrees with other result of geometric discord based on weak measurements that is always smaller than normal geometric discord. The next chapter is devoted to present our definition of quantum discord based on weak measurements, and show how to link it with a global and local reality as induced by weak measurements. Also, we discuss how we interpret this quantity based on the amount of quantum correlations that is destroyed by local weak measurements.

5 WEAK QUANTUM DISCORD

The results and most of the content of this chapter are published in [P. R. Dieguez and R. M. Angelo, Quantum Inf. Processing **17** 194 (2018)] [64].

5.1 WEAK QUANTUM DISCORD

Another contribution of this thesis is in the direction of how to properly define and interpret the quantifiers that measure quantum correlation if we take a perspective of an arbitrary intensity measurement. Singh and Pati showed that quantum discord results in higher values when projective measurements are substituted by weak measurements, which sounds paradoxical since weaker measurements should imply weaker disturbance and, thus, a smaller distance in a distance-based formulation. We propose to solve this puzzle by presenting a quantifier and an underlying interpretation for the weak quantum discord [64]. It is possible to show that if we take a distance-based formulation as a primitive notion for quantum discord, as pondered in Refs. [72, 84], then no surprise is found when replacing projective measurements with weak ones. In particular, no "super" quantum discord emerges. As we show in what follows, the weak quantum discord interpolates between the regime of "no quantum correlations destroyed" (when no measurement is conducted) and the regime of "all quantum correlations destroyed" (when a projective measurement is conducted), in which case the quantum discord is recovered.

To investigate what happens with quantum discord as the projective measurements are replaced with weak measurements, Singh and Pati [89] employed the weak-measurement dichotomic operators introduced by Oreshkov and Brun [87], namely,

$$P_{\pm}(x) = \sqrt{\frac{1 \mp \tanh x}{2}}\Pi_0 + \sqrt{\frac{1 \pm \tanh x}{2}}\Pi_1, \tag{5.1}$$

with $x \in \mathbb{R}$ and $\Pi_0 + \Pi_1 = P_+^2 + P_-^2 = \mathbb{1}$ for projectors Π_0 and Π_1 acting on $\mathcal{H}_{\mathcal{B}}$. Singh and Pati then used these operators to construct the post-measurement state

$$\rho_{\mathcal{A}|P_\pm} = \mathrm{Tr}_{\mathcal{B}}[(\mathbb{1}\otimes P_\pm)\rho(\mathbb{1}\otimes P_\pm)]/p_\pm, \tag{5.2}$$

with probabilities $p_\pm = \mathrm{Tr}[(\mathbb{1}\otimes P_\pm)\rho(\mathbb{1}\otimes P_\pm)]$, and the "weak conditional entropy"

$$S_x(\mathcal{A}|\{P_\pm\}) = p_+ S(\rho_{\mathcal{A}|P_+}) + p_- S(\rho_{\mathcal{A}|P_-}). \tag{5.3}$$

With that, they introduced the *super quantum discord* (SQD)

$$D^x_{\mathcal{B}}(\rho) = \min_{\{P_\pm\}} \sum_{s=\pm} p_s S(\rho_{\mathcal{A}|P_s}) + S(\rho_{\mathcal{B}}) - S(\rho), \tag{5.4}$$

which is a function of x. The name indeed is appropriate as Singh and Pati have proved that $D^x_{\mathcal{B}}(\rho) \geqslant D_{\mathcal{B}}(\rho)$. Although the formula (5.4) for the SQD is a natural generalization of the expression (2.58) for the QD, it produces a conflict with the intuition deriving from the alternative form (2.61): A weak measurement should imply a weak disturbance on the measured state and, therefore, a weak discord instead of a super discord. In particular, for $x = 0$ we have $P_\pm(0) = \mathbb{1}/\sqrt{2}$, which should imply no change in the state. Still, from the formula (5.4) we obtain $D^{x=0}_{\mathcal{B}}(\rho) = I_{\mathcal{A}:\mathcal{B}}(\rho)$, which is clearly non-zero.

Let us compute the QD using the Singh and Pati procedure, which consists of replacing the original form (2.58) with (5.4), but now using our weak-measurement map (4.2). We find

$$\mathfrak{D}^\epsilon_{\mathcal{B}}(\rho) = \min_B \sum_b p_b S(C^\epsilon_{b|B}(\rho)) + S(\rho_{\mathcal{B}}) - S(\rho). \tag{5.5}$$

It follows from the concavity and the additivity of the von Neumann entropy that

$$\begin{aligned}
\mathfrak{D}^\epsilon_{\mathcal{B}}(\rho) &> \min_B \sum_b p_b \Big[(1-\epsilon)S(\rho) + \epsilon S(\rho_{\mathcal{A}|b})\Big] + S(\rho_{\mathcal{B}}) - S(\rho) \\
&= (1-\epsilon)S(\rho) + \epsilon \min_B \sum_b p_b S(\rho_{\mathcal{A}|b}) + S(\rho_{\mathcal{B}}) - S(\rho) + \epsilon\Big[S(\rho_{\mathcal{B}}) - S(\rho_{\mathcal{B}})\Big] \\
&= \epsilon\Big[\min_B \sum_b p_b S(\rho_{\mathcal{A}|b}) + S(\rho_{\mathcal{B}}) - S(\rho)\Big] + (1-\epsilon)S(\rho_{\mathcal{B}}) \\
&= \epsilon D_{\mathcal{B}}(\rho) + (1-\epsilon)S(\rho_{\mathcal{B}}).
\end{aligned} \tag{5.6}$$

Since

$$\mathfrak{D}_{\mathcal{B}}^{\epsilon\to 0}(\rho) > S(\rho_{\mathcal{B}}), \tag{5.7}$$

the drawback of Singh and Pati's approach persists, that is, the definition (5.5) is not able as well to predict, for any ρ, zero distance in the limit of no disturbance.

Now we change the strategy. Given that the forms (2.58) and (2.61) are mathematically equivalent upon the use of projective measurements, one might think at a first sight that there is no reason *a priori* for one to prefer one of them when weak measurements are used instead. However, we should realize that a weak measurement does not provide a precise outcome on which we could apply the *conditioning*, so that the meaning of the form (2.58), which is based on the conditional entropy, becomes unclear in this case. We then take the form (2.61) as the primitive notion of QD. In terms of the monitoring (4.20), this allows us to introduce the *weak quantum discord* (WQD):

$$\mathcal{D}_{\mathcal{B}}{}^{\epsilon}(\rho) := \min_{B} \left[I_{\mathcal{A}:\mathcal{B}}(\rho) - I_{\mathcal{A}:\mathcal{B}}(\mathcal{M}_B^{\epsilon}(\rho)) \right] \qquad (0 < \epsilon < 1), \tag{5.8}$$

which clearly reduces to QD as $\epsilon \to 1$. Most importantly, this form trivially implements the feature we have been looking for, namely,

$$\mathcal{D}_{\mathcal{B}}{}^{\epsilon\to 0}(\rho) = 0 \ (\forall \rho). \tag{5.9}$$

We now prove a result that precisely defines the sense in which the quantum discord quantifier $\mathcal{D}_{\mathcal{B}}{}^{\epsilon}$ can be termed genuinely *weak*.

Teorema 5.1.1. *For any density operator ρ on $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$ and ϵ real such that $\epsilon \in (0,1)$, the weak quantum discord* (5.8) *is never greater than the quantum discord* (2.61)*, that is, $\mathcal{D}_{\mathcal{B}}{}^{\epsilon}(\rho) \leqslant D_{\mathcal{B}}(\rho)$. The equality holds for quantum-classical states of the form $\Phi_B(\rho) = \rho$, in which case $\mathcal{D}_{\mathcal{B}}{}^{\epsilon} = D_{\mathcal{B}} = 0$.*

Proof.—Consider an instance in which a system $\mathcal{AB}$ initially prepared in a density operator ρ on $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$ ends up into $M_B^{\epsilon}(\rho)$ after the monitoring of a generic observable B on $\mathcal{H}_{\mathcal{B}}$. The Stinespring theorem [59] ensures that this mapping can be cast in terms of an entangling dynamics $U(t)$ between $\mathcal{B}$ and some extra degree of freedom $\mathcal{X}$ initially prepared in a state $|x_0\rangle\langle x_0|$, that is,

$$\mathcal{M}_B^\epsilon(\rho) = \mathrm{Tr}_{\mathcal{X}}\left[U(t)\,\rho \otimes |x_0\rangle\langle x_0|\,U^\dagger(t)\right] = \rho_{\mathcal{AB}}(t), \tag{5.10}$$

with $U(t)$ acting on $\mathcal{H}_{\mathcal{B}} \otimes \mathcal{H}_{\mathcal{X}}$. By direct application of the partial trace we obtain

$$\mathrm{Tr}_{\mathcal{A}}[\mathcal{M}_B^\epsilon(\rho)] = \mathcal{M}_B^\epsilon(\rho_{\mathcal{B}}) = \rho_{\mathcal{B}}(t). \tag{5.11}$$

In addition, $\rho_{\mathcal{BX}}(t) = U(t)\,\rho_{\mathcal{B}} \otimes |x_0\rangle\langle x_0|\,U^\dagger(t)$ and $\rho_{\mathcal{A}}(t) = \rho_{\mathcal{A}}$. The unitary invariance of the von Neumann entropy allows us to write $S(\rho_{\mathcal{ABX}}(t)) = S(\rho)$ and $S(\rho_{\mathcal{BX}}(t)) = S(\rho_{\mathcal{B}})$. From the strong subadditivity of the von Neumann entropy, $S(\rho_{\mathcal{ABX}}(t)) + S(\rho_{\mathcal{B}}(t)) \leqslant S(\rho_{\mathcal{AB}}(t)) + S(\rho_{\mathcal{BX}}(t))$ [59], one then obtains

$$S(\rho) + S(\mathcal{M}_B^\epsilon(\rho_{\mathcal{B}})) \leqslant S(\mathcal{M}_B^\epsilon(\rho)) + S(\rho_{\mathcal{B}}). \tag{5.12}$$

Since $\mathcal{M}_B^\epsilon(\rho_{\mathcal{A}}) = \rho_{\mathcal{A}}$, it immediately follows from the definition of mutual information that

$$I_{\mathcal{A}:\mathcal{B}}(\rho) \geqslant I_{\mathcal{A}:\mathcal{B}}(\mathcal{M}_B^\epsilon(\rho)). \tag{5.13}$$

This is a statement of the monotonicity of the mutual information under unrevealed weak measurements—an expected result since monitoring, as defined by the completely-positive trace preserving map (4.20), is, after all, a quantum operation [36]. Also, this proves that the WQD is non-negative. Since $\mathcal{M}_B^\epsilon \Phi_B(\rho) = \Phi_B(\rho)$, it can be directly checked that the equality holds for $\rho = \Phi_B(\rho) = \sum_b p_b \rho_{\mathcal{A}|b} \otimes B_b$, that is, when the preparation ρ is a quantum-classical state (a state of reality for the observable B [53]). In this case, both the WQD and the QD vanish[1]. We now employ the property

$$[\mathcal{M}_B^\epsilon]^n(\rho) = (1-\epsilon)^n \rho + [1 - (1-\epsilon)^n]\Phi_B(\rho), \tag{5.14}$$

which has been proved in Ref. [36] for the map (4.20) and from which we can directly show that $[\mathcal{M}_B^\epsilon]^{n\to\infty}(\rho) = \Phi_B(\rho)$. Via successive application of the relation (5.13) we

[1]Of course the WQD also vanishes for $\epsilon \to 0$, but this trivial limit is not included in the statement of the Theorem 1.

obtain

$$I_{\mathcal{A}:\mathcal{B}}(\mathcal{M}_B^{\epsilon}(\rho)) \geqslant I_{\mathcal{A}:\mathcal{B}}([\mathcal{M}_B^{\epsilon}]^2(\rho)) \geqslant \cdots \geqslant I_{\mathcal{A}:\mathcal{B}}(\Phi_B(\rho)). \tag{5.15}$$

It then follows that

$$\mathcal{D}_{\mathcal{B}}{}^{\epsilon}(\rho) = \min_B \Big[I_{\mathcal{A}:\mathcal{B}}(\rho) - I_{\mathcal{A}:\mathcal{B}}(\Phi_B(\rho)) + I_{\mathcal{A}:\mathcal{B}}(\Phi_B(\rho)) - I_{\mathcal{A}:\mathcal{B}}(\mathcal{M}_B^{\epsilon}(\rho)) \Big] \leqslant D_{\mathcal{B}}(\rho), \tag{5.16}$$

which completes the proof. ■

To emphasize the issue around the SQD, a remark is in order. Let us consider an unrevealed weak-measurement map $\Phi_{\{P_\pm\}}(\rho) = \sum_s P_s \rho P_s$ composed of the dichotomic operators (3.31). Since $\Pi_{0(1)} = \mathbb{1} - \Pi_{1(0)}$ one shows, by direct manipulation, that

$$\Phi_{\{P_\pm\}}(\rho) = \mathrm{sech}(x)\,\rho + \big[1 - \mathrm{sech}(x)\big]\,\Phi_\Pi(\rho) = M_\Pi^{1-\mathrm{sech}(x)}(\rho), \tag{5.17}$$

which holds for all x and for the operator $\Pi = \sum_s \pi_s \Pi_s$. In fact, consider that

$$P(x) = t_0 P_0 + t_1 P_1 \tag{5.18}$$

and

$$P(-x) = t_1 P_0 + t_0 P_1, \tag{5.19}$$

with $t_0 = \sqrt{\frac{1-\tanh x}{2}}$, $t_1 = \sqrt{\frac{1+\tanh x}{2}}$ and $P_0 + P_1 = \mathbb{1}$. Also, $t_0 t_1 = \frac{\sqrt{1-\tanh^2 x}}{2}$ and $t_0^2 + t_1^2 = 1$ we have that:

$$\Phi_{P(\pm x)}(\rho) := (t_0 P_0 + t_1 P_1)\rho(t_0 P_0 + t_1 P_1) + (t_1 P_0 + t_0 P_1)\rho(t_1 P_0 + t_0 P_1), \tag{5.20}$$

with

$$(t_0 P_0 + t_1 P_1)\rho(t_0 P_0 + t_1 P_1) = t_0 P_0 \rho t_0 P_0 + t_0 P_0 \rho t_1 P_1 + t_1 P_1 \rho t_0 P_0 + t_1 P_1 \rho t_1 P_1 \tag{5.21}$$

and

$$(t_1 P_0 + t_0 P_1)\rho(t_1 P_0 + t_0 P_1) = t_1 P_0 \rho t_1 P_0 + t_1 P_0 \rho t_0 P_1 + t_0 P_1 \rho t_1 P_0 + t_0 P_1 \rho t_0 P_1. \tag{5.22}$$

Note that $t_0P_0\rho t_0P_0 + t_0P_1\rho t_0P_1 = t_0^2\Phi_P(\rho)$ and $t_1P_1\rho t_1P_1 + t_1P_0\rho t_1P_0 = t_1^2\Phi_P(\rho)$. With that we can write

$$\Phi_{P(\pm x)}(\rho) = \Phi_P(\rho) + 2t_0t_1(P_0\rho P_1 + P_1\rho P_0) \tag{5.23}$$

and using $P_0 + P_1 = \mathbb{1}$, it follows that

$$\Phi_{P(\pm x)}(\rho) = \Phi_P(\rho) + 2t_0t_1(\rho P_1 + \rho P_0) - 2t_0t_1(P_0\rho P_0 + P_1\rho P_1) \tag{5.24}$$

and,

$$\Phi_{P(\pm x)}(\rho) = 2t_0t_1\rho + (1 - 2t_0t_1)\Phi_P(\rho). \tag{5.25}$$

Finally, we arrive at

$$\Phi_{P(\pm x)}(\rho) = \mathrm{sech}(x)\,\rho + \big[1 - \mathrm{sech}(x)\big]\Phi_\Pi(\rho). \tag{5.26}$$

This shows that the dichotomic map $\Phi_{\{P_\pm\}}$ is a specialization of the map M_B^ϵ and, as such, definitively allows for a proper definition of WQD in the molds of the proposal (5.8). Hence, the conditioning to undefined outcomes turns out to be the only conceptual difficulty associated to the SQD proposal.

5.1.1 Interpretation

The fact that the distance-based formulation (5.8) leads to ${\mathcal{D}_\mathcal{B}}^{\epsilon\to 0}(\rho) = 0$ even for discordant states, that is, states for which $D_\mathcal{B}(\rho) > 0$, raises the question as to whether the WQD can be viewed as a faithful quantifier of quantum correlations. We now point out the precise meaning that we propose to attach to the WQD. Let us consider the measure

$$\mathfrak{a}(\rho, \sigma) = I_{\mathcal{A}:\mathcal{B}}(\rho) - I_{\mathcal{A}:\mathcal{B}}(\sigma) \tag{5.27}$$

for any ρ and σ on $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$. With that, QD can be written as

$$D_\mathcal{B}(\rho) = \min_B\ \mathfrak{a}(\rho, \Phi_B(\rho)) = \mathfrak{a}(\rho, \Phi_{B_1}(\rho)), \tag{5.28}$$

where B_1 is the observable that implements the minimization. Using the traditional interpretation of QD we then take $\mathfrak{a}(\rho, \Phi_{B_1}(\rho))$ as the amount of quantum correlations that can be associated to ρ under local measurements of the optimal observable B_1. For the WQD we similarly write

$$\mathcal{D}_{\mathcal{B}}{}^{\epsilon}(\rho) = \min_{B} \ \mathfrak{a}(\rho, \mathcal{M}_B^{\epsilon}(\rho)) = \mathfrak{a}(\rho, \mathcal{M}_{B_\epsilon}^{\epsilon}(\rho)), \tag{5.29}$$

where B_ϵ denotes the (ϵ-dependent) optimal observable.

Now, consider the state $\tilde{\rho}_\epsilon = \mathcal{M}_{B_\epsilon}^{\epsilon}(\rho)$, which refers to a scenario in which a preparation ρ has undergone a monitoring of the observable B_ϵ. Since $\Phi_B \mathcal{M}_B^{\epsilon}(\rho) = \Phi_B(\rho)$ for all B, then $\Phi_{B_\epsilon}(\tilde{\rho}) = \Phi_{B_\epsilon}(\rho)$. Via

$$I_{\mathcal{A}:\mathcal{B}}(\rho) - I_{\mathcal{A}:\mathcal{B}}(\mathcal{M}_B^{\epsilon}(\rho)) = I_{\mathcal{A}:\mathcal{B}}(\rho) - I_{\mathcal{A}:\mathcal{B}}(\Phi_B(\rho)) - (I_{\mathcal{A}:\mathcal{B}}(\mathcal{M}_B^{\epsilon}(\rho) - I_{\mathcal{A}:\mathcal{B}}(\Phi_B \mathcal{M}_B^{\epsilon}(\rho))), \tag{5.30}$$

we have

$$\mathcal{D}_{\mathcal{B}}{}^{\epsilon}(\rho) = \mathfrak{a}(\rho, \Phi_{B_\epsilon}(\rho)) - \mathfrak{a}(\tilde{\rho}_\epsilon, \Phi_{B_\epsilon}(\tilde{\rho}_\epsilon)), \tag{5.31}$$

which settles the interpretation for the WQD. The first term on right-hand side of Eq. (5.31) refers to the amount of quantum correlations encoded in ρ, whereas the second encapsulates the amount of quantum correlations that persist after the monitoring of the optimal observable B_ϵ. Thus, $\mathcal{D}_{\mathcal{B}}{}^{\epsilon}(\rho)$ can be viewed as *the amount of quantum correlations that is removed from ρ by local weak measurements of B_ϵ*. In consonance with this interpretation, we see for $\epsilon \to 0$ that the above formula readily gives

$$\mathcal{D}_{\mathcal{B}}{}^{\epsilon\to 0}(\rho) = 0, \tag{5.32}$$

meaning that no quantum correlation is destroyed when no measurement is performed. On the other hand, for $\epsilon \to 1$, the second term of Eq. (5.31) vanishes and we find

$$\mathcal{D}_{\mathcal{B}}{}^{\epsilon\to 1}(\rho) = \mathfrak{a}(\rho, \Phi_{B_1}(\rho)) = D_{\mathcal{B}}(\rho), \tag{5.33}$$

meaning that all quantum correlations can be destroyed via projective measurements of the optimal observable.

5.1.2 Example

Let us consider the one-parameter state of two qubits:

$$\rho^{\mu} = (1-\mu)\frac{\mathbb{1}\otimes\mathbb{1}}{4} + \mu|s\rangle\langle s|, \tag{5.34}$$

with the singlet state

$$|s\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \tag{5.35}$$

Using the definition of mutual information, we have

$$I_{\mathcal{A}:\mathcal{B}}(\rho^{\mu}) - I_{\mathcal{A}:\mathcal{B}}(\mathcal{M}_B^{\epsilon}(\rho^{\mu})) = S(\rho^{\mu}_{\mathcal{B}}) - S(\rho^{\mu}) - S(\mathcal{M}_B^{\epsilon}\rho^{\mu}_{\mathcal{B}}) + S(\mathcal{M}_B^{\epsilon}\rho^{\mu}). \tag{5.36}$$

Noticing that

$$\rho^{\mu}_{\mathcal{B}} = \mathcal{M}_B^{\epsilon}(\rho^{\mu}_{\mathcal{B}}) = \mathbb{1}/2 \tag{5.37}$$

reduces the WQD (5.8) to

$$\mathcal{D}_{\mathcal{B}}{}^{\epsilon}(\rho^{\mu}) = \min_{B}\Big[S(\mathcal{M}_B^{\epsilon}(\rho^{\mu})) - S(\rho^{\mu})\Big]. \tag{5.38}$$

The eigenvalues of ρ^{μ} are given by $\{\frac{1-\mu}{4}, \frac{1-\mu}{4}, \frac{1-\mu}{4}, \frac{1+3\mu}{4}\}$. To compute the eigenvalues of $\mathcal{M}_B^{\epsilon}(\rho^{\mu})$ we introduce the generic observable $B = \sum_{b=\pm} bB_b$ with projectors $B_{\pm} = |\pm\rangle\langle\pm|$ such that

$$|+\rangle = \cos\left(\tfrac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\tfrac{\theta}{2}\right)|1\rangle \tag{5.39}$$

and

$$|-\rangle = -\sin\left(\tfrac{\theta}{2}\right)|0\rangle + e^{i\phi}\cos\left(\tfrac{\theta}{2}\right)|1\rangle. \tag{5.40}$$

Then we can directly compute

$$\Phi_B(\rho^{\mu}) = \sum_s (\mathbb{1}\otimes B_s)\rho^{\mu}(\mathbb{1}\otimes B_s) \tag{5.41}$$

and the weakly measured state

$$\mathcal{M}_B^{\epsilon}(\rho^{\mu}) = (1-\epsilon)\rho^{\mu} + \epsilon\Phi_B(\rho^{\mu}), \tag{5.42}$$

whose eigenvalues can be shown to be $\{\frac{1-\mu}{4}, \frac{1-\mu}{4}, \frac{1+3\mu-2\mu\epsilon}{4}, \frac{1-\mu+2\mu\epsilon}{4}\}$. As a consequence of the rotational invariance of the singlet state, this set has no information about the parameters (θ, ϕ), which would be used for the minimization process. With the pertinent eigenvalues at hand, we can evaluate the entropies in Eq. (5.38) and then finally write the WQD in compact form as

$$\mathcal{D}_{\mathcal{B}}{}^{\epsilon}(\rho^{\mu}) = \tfrac{1}{4}\sum_{i=-1}^{1}\sum_{j=0}^{1}(-1)^j \lambda_{ij}\ln\lambda_{ij}, \qquad \lambda_{ij} = 1+\mu[1+2i(1-j\epsilon)]. \tag{5.43}$$

This function is plotted in Fig. 5.1, where we can see that it indeed has the behavior expected for a genuine WQD: it is always lower than QD, as implied by Theorem 1, that is,

$$\mathcal{D}_{\mathcal{B}}{}^{\epsilon}(\rho^{\mu}) < \mathcal{D}_{\mathcal{B}}{}^{\epsilon\to 1}(\rho^{\mu}) = D_{\mathcal{B}}(\rho^{\mu}) \tag{5.44}$$

$\forall\epsilon \in (0,1)$, and disappears for a vanishing monitoring

$$\mathcal{D}_{\mathcal{B}}{}^{\epsilon\to 0}(\rho^{\mu}) = 0. \tag{5.45}$$

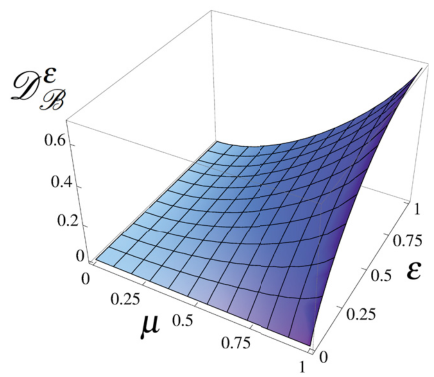

Figure 5.1: Weak quantum discord $\mathcal{D}_{\mathcal{B}}{}^{\epsilon}(\rho^{\mu})$, as given by Eq. (5.43), for the state ρ^{μ} as a function of μ and the strengh ϵ of the measurement. This is an illustration of Theorem 1, since $\mathcal{D}_{\mathcal{B}}{}^{\epsilon}(\rho^{\mu}) < \mathcal{D}_{\mathcal{B}}{}^{\epsilon\to 1}(\rho^{\mu}) = D_{\mathcal{B}}(\rho^{\mu})$ $\forall\epsilon \in (0,1)$. In particular $\mathcal{D}_{\mathcal{B}}{}^{\epsilon\to 0}(\rho^{\mu}) = 0$.

5.2 SYMMETRICAL WEAK QUANTUM DISCORD

In Ref. [72], Rulli and Sarandy defined the *symmetrical quantum discord* (SyQD)

$$D(\rho) = \min_{A,B} \Big[I_{\mathcal{A}:\mathcal{B}}(\rho) - I_{\mathcal{A}:\mathcal{B}}(\Phi_A \Phi_B(\rho)) \Big], \tag{5.46}$$

for observables $A = \sum_a aA_a$ and $B = \sum_b bB_b$ acting on $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$, respectively, projectors $\{A_a, B_b\}$, and the map

$$\Phi_A(\rho) = \sum_a (A_a \otimes \mathbb{1})\rho(A_a \otimes \mathbb{1}) \tag{5.47}$$

in analogy with the map (2.60). This quantifier is "symmetrical" in that both parties of the system are measured. Inspired by this definition, we introduce

$$\mathcal{D}^{(\epsilon',\epsilon)}(\rho) := \min_{A,B} \Big[I_{\mathcal{A}:\mathcal{B}}(\rho) - I_{\mathcal{A}:\mathcal{B}}(\mathcal{M}_A^{\epsilon'} \mathcal{M}_B^{\epsilon}(\rho)) \Big] \qquad (0 < \{\epsilon', \epsilon\} < 1), \tag{5.48}$$

as a quantifier of *symmetrical weak quantum discord* (SyWQD). The monitoring of the observable A is given by

$$\mathcal{M}_A^{\epsilon'} = (1-\epsilon')\rho + \epsilon' \Phi_A(\rho) \tag{5.49}$$

in analogy with the monitoring (4.20). From the monotonicity of the mutual information [see relation (5.13)], it follows that SyWQD is a non-negative quantity. Also, $\mathcal{D}^{(\epsilon',\epsilon)}(\rho)$ will be zero only if

$$\rho = \Phi_A \Phi_B(\rho) = \sum_{a,b} p_{ab} A_a \otimes B_b, \tag{5.50}$$

that is, for a state with no SyQD (a classical-classical state)[2]. In light of the interpretation proposed in Sec. 5.1.1, we claim that the SyWQD should be viewed as a measure of the amount of quantum correlations that is removed form the state ρ by bi-local weak measurements. This position is supported by the fact that

$$\mathcal{D}^{(\epsilon',\epsilon)\to(0,0)}(\rho) = 0 \tag{5.51}$$

[2]Of course, the SyWQD will also vanishes for $(\epsilon', \epsilon) \to (0,0)$.

and

$$\mathcal{D}^{(\epsilon',\epsilon)\to(1,1)}(\rho) = D(\rho). \tag{5.52}$$

5.2.1 Hierarchy and Ordering

Interestingly, by use of the monotonicity of the mutual information (5.13) and the procedures employed to prove Theorem 1 [see formulas (5.13)-(5.16)] we can make some statements about ordering and hierarchy of discord measures. By adding and subtracting $I_{\mathcal{A}:\mathcal{B}}(\Phi_B(\rho))$ in Eq. (5.46), we can prove that $D \geqslant D_{\mathcal{B}}$ (of course, it also holds that $D \geqslant D_{\mathcal{A}}$). This relation defines a *hierarchy*, in the same sense as discussed in Refs. [55, 83]. This means that here we have a specific direction of implication between two quantities that are conceptually different. To appreciate this point, consider a quantum-classical state $\rho = \Phi_B(\rho)$. While for this state the QD is zero, the SyQD may not be. That is, the SyQD quantifies correlations that cannot be destroyed solely by measurements of B. It follows that not all symmetrically discordant state (those with $D > 0$) are discordant states (those for which $D_{\mathcal{B}} > 0$), while the converse is true. In other words, discordant states form a subset of symmetrically discordant states, so that the detection of QD for a given state immediately implies the existence of SyQD for this state. By introducing $I_{\mathcal{A}:\mathcal{B}}(M_B^{\epsilon}(\rho))$ in the definition (5.48) one finds $\mathcal{D}^{(\epsilon',\epsilon)} \geqslant \mathcal{D}_{\mathcal{B}}{}^{\epsilon}$ (and, analogously, $\mathcal{D}^{(\epsilon',\epsilon)} \geqslant \mathcal{D}_{\mathcal{A}}^{\epsilon}$), which shows that an equivalent hierarchy applies for the corresponding weak quantifiers.

On the other hand, let us similarly introduce $I_{\mathcal{A}:\mathcal{B}}(\Phi_A\Phi_B(\rho))$ in the Eq. (5.48) and use the monotonicity again. In this case we arrive at $\mathcal{D}^{(\epsilon',\epsilon)} \leqslant D$, which is a mere statement of *ordering*. That is, a simple comparison relation between two quantities that identify the same class of quantum correlations, namely, those that are destroyed by local monitorings in the sites $\mathcal{A}$ and $\mathcal{B}$. In fact, notice that both the SyWQD and SyQD vanish only for classical-classical states of the form $\rho = \Phi_A\Phi_B(\rho)$, thus meaning that set of symmetrically discordant states (those for which $D > 0$) and the set of symmetrically weakly discordant states (those with $\mathcal{D}^{(\epsilon',\epsilon)} > 0$) are one and the same. Similar conclusions apply for the quantifiers appearing in the Theorem 1.

5.3 WEAK QUANTUM DICORD AND WEAK REALITY

Now, we return to the initial question regarding the relation between quantum correlations and reality change in the weak measurement regime. Consider the nonminimized version of weak quantum discord, and notice, via definition 4.37, that

$$\begin{aligned}
\mathcal{D}_{\mathcal{B}}{}^{\epsilon}(\rho) =& I_{\mathcal{A}:\mathcal{B}}(\rho) - I_{\mathcal{A}:\mathcal{B}}(M_B^{\epsilon}(\rho)) && (5.53)\\
=& S(\rho_{\mathcal{A}}) + S(\rho_{\mathcal{B}}) - S(\rho) - S(\rho_{\mathcal{A}}) - S(M_B^{\epsilon}(\rho_{\mathcal{B}})) + S(M_B^{\epsilon}(\rho)) && (5.54)\\
=& S(M_B^{\epsilon}(\rho)) - S(\rho) - (S(M_B^{\epsilon}(\rho_{\mathcal{B}})) - S(\rho_{\mathcal{B}}))\\
=& \Delta\mathfrak{R}(B)_{\rho} - \Delta\mathfrak{R}(B)_{\rho_{\mathcal{B}}},
\end{aligned}$$

that is,

$$\Delta\mathfrak{R}(B)_{\rho} = \Delta\mathfrak{R}(B)_{\rho_{\mathcal{B}}} + \mathcal{D}_{\mathcal{B}}{}^{\epsilon}(\rho). \tag{5.55}$$

Now, it becomes clear that the degree of variation in reality of B is the sum of the degree of local variation of reality (that is, the variation of B given the reduced state $\rho_{\mathcal{B}}$) with weak correlations associated with measurements of B.

Before we definitively assert that it is a genuine measure of quantum correlations, an important question is to be answered. It is well known that quantum discord reduces to the entanglement entropy for pure states, that is, $D_{\mathcal{B}}(|\psi\rangle) = S(\rho_{\mathcal{A}(\mathcal{B})})$, for reduced states $\rho_{\mathcal{A}(\mathcal{B})} = \mathrm{Tr}_{\mathcal{B}(\mathcal{A})}|\psi\rangle\langle\psi|$. This can be proved by taking B as the operator whose projectors $|b_i\rangle\langle b_i|$ define the Schmidt decomposition $|\psi\rangle = \sum_i \sqrt{\lambda_i}|a_i\rangle|b_i\rangle$. By virtue of Theorem 1 we can directly conclude that $\mathcal{D}_{\mathcal{B}}{}^{\epsilon}(|\psi\rangle) \leqslant S(\rho_{\mathcal{A}(\mathcal{B})})$, but this does not allows us to claim that $\mathcal{D}_{\mathcal{B}}{}^{\epsilon}(|\psi\rangle)$ is an entanglement monotone. To this end one should be able to prove that $\mathcal{D}_{\mathcal{B}}{}^{\epsilon}(|\psi\rangle)$ does not increase on average under local measurements and classical communication [118]. Actually, to be fair, the very weak quantum discord (along with all its related quantifiers) is to be submitted to some reliability criteria [119].

Equipped with all the tools presented so far, we will now focus on revisiting some foundational questions in QM, such as the measurement problem, Everett's paradox, Wigner's paradox, and the well-known wave-particle duality.

6 INFORMATION-REALITY APPROACH TO QUANTUM ISSUES

The results and content of sections 6.1 and 6.2 of this chapter are published in [P. R. Dieguez and R. M. Angelo, Phys. Rev. A **97**, 022107 (2018)] [36].

6.1 EVERETT'S PARADOX

The foundational relevance of the measurement problem needs no emphasis. Here we hope to shed some light on this longstanding issue by using the tools introduced above. We consider the well-known drama proposed by Everett [113], in which an external observer describes a measurement conducted within a laboratory by an internal observer. The conflict emerges as we note that for the internal observer an irreversible state reduction occurs, whereas for the external one, who can conceive of the internal observer as part of a physical system, only a reversible dynamics takes place.

To approach this puzzle we take an informational perspective and consider, from the viewpoint of the external observer $\mathcal{O}_{\text{ext}}$, three physical systems, namely, the internal observer $\mathcal{O}$, an apparatus $\mathcal{A}$, and a system of interest $\mathcal{S}$. These systems are described quantum mechanically by $\mathcal{O}_{\text{ext}}$, who naturally does not include oneself in the description. Let $\sigma_{\mathcal{AS}}$ be the joint state after the apparatus has got correlated with the system. In the last stage of the measurement process, $\mathcal{O}$ looks at the apparatus, that is, indirectly interacts with $\mathcal{A}$ by means of the photons scattered by $\mathcal{A}$. Without interacting with the joint system $\mathcal{OAS}$, $\mathcal{O}_{\text{ext}}$ describes the dynamics in terms of the unitary evolution

$$\rho_{\mathcal{OAS}} = U_{\mathcal{OA}}\left(\sigma_{\mathcal{AS}} \otimes |o\rangle\langle o|\right) U_{\mathcal{OA}}^{\dagger}. \tag{6.1}$$

According to Eq. (3.46), the information is distributed over the system as

$$I(\rho_{\mathcal{OAS}}) = I(\rho_{\mathcal{O}}) + I_{\mathcal{O}:\mathcal{AS}}(\rho_{\mathcal{OAS}}) + I(\rho_{\mathcal{AS}}). \tag{6.2}$$

The first two terms on the right-hand side refer to information that cannot be accessed by $\mathcal{O}$, as they refer to the state of $\mathcal{O}$ and its correlation with the part $\mathcal{AS}$ as seen from the perspective of $\mathcal{O}_{\text{ext}}$. This is an irremovable limitation because $\mathcal{O}$ cannot ascribe a quantum state for oneself and therefore has no way to assess the terms $I(\rho_{\mathcal{O}}) + I_{\mathcal{O}:\mathcal{AS}}(\rho_{\mathcal{OAS}})$. Let us doubly emphasize this point by recalling that no reference frame can describe its own physical state. By its turn, the third term on the right-hand side can be written, according to Eq. (3.54), as

$$I(\rho_{\mathcal{AS}}) = I(\rho_{\mathcal{A}}) + I_{\mathcal{S}|\mathcal{A}}(\rho_{\mathcal{AS}}), \tag{6.3}$$

where $I_{\mathcal{S}|\mathcal{A}}$ is expected to be the only informational content that $\mathcal{O}$ can obtain about $\mathcal{S}$ through the measurement process. To see that this is indeed the case, let us move to $\mathcal{O}$'s reference frame, wherein the unitary evolution $U_{\mathcal{OA}}$ is not applicable. According to the reduction postulate, upon collection of scattered photons, $\mathcal{O}$ will (somehow) perceive a state $\mathcal{C}_{a|A}(\sigma_{\mathcal{AS}}) = A_a \otimes \sigma_{\mathcal{S}|a}$ in a particular run of the experiment. The average entropy associated with many runs will be

$$\bar{S}_{\mathcal{S}|\mathcal{A}} = \sum_a p_a S(\mathcal{C}_{a|A}(\sigma_{\mathcal{AS}})) = \sum_a p_a S(\sigma_{\mathcal{S}|a}). \tag{6.4}$$

Using the joint entropy theorem [59], one shows that this result can be written as

$$\bar{S}_{\mathcal{S}|\mathcal{A}} = S(\Phi_A(\sigma_{\mathcal{AS}})) - S(\Phi_A(\sigma_{\mathcal{A}})) = S_{\mathcal{S}|\mathcal{A}}(\Phi_A(\sigma_{\mathcal{AS}})), \tag{6.5}$$

which refers to the remaining ignorance about $\mathcal{S}$ given that $\mathcal{A}$ has been accessed and collapsed. The average information acquired by $\mathcal{O}$ about $\mathcal{S}$ through the observation of $\mathcal{A}$ is, by definition, $\bar{I}_{\mathcal{S}|\mathcal{A}} := \ln d_{\mathcal{S}} - \bar{S}_{\mathcal{S}|\mathcal{A}}$. It can be written as

$$\bar{I}_{\mathcal{S}|\mathcal{A}} = \ln d_{\mathcal{S}} - \sum_a p_a S(\mathcal{C}_{a|A}(\sigma_{\mathcal{AS}})). \tag{6.6}$$

To compute $I_{\mathcal{S}|\mathcal{A}}$, the information that $\mathcal{O}$ can access about $\mathcal{S}$ via interaction with $\mathcal{A}$, from $\mathcal{O}_{\text{ext}}$'s perspective, we apply the Stinespring theorem to write

$$\rho_{\mathcal{AS}} = \text{Tr}_{\mathcal{O}}\rho_{\mathcal{OAS}} = \Phi_A(\sigma_{\mathcal{AS}}), \tag{6.7}$$

which presumes that a strong monitoring has occurred inside $\mathcal{O}$'s laboratory. It follows from the definition of conditional information that

$$I_{\mathcal{S}|\mathcal{A}} = \ln d_{\mathcal{S}} - S_{\mathcal{S}|\mathcal{A}}(\Phi_A(\sigma_{\mathcal{AS}})) = \ln d_{\mathcal{S}} - \bar{S}_{\mathcal{S}|\mathcal{A}}, \tag{6.8}$$

which implies that $I_{\mathcal{S}|\mathcal{A}} = \bar{I}_{\mathcal{S}|\mathcal{A}}$, as we wanted to prove. This result can also be written as

$$I_{\mathcal{S}|\mathcal{A}}(\Phi_A(\sigma_{\mathcal{AS}})) = \ln d_{\mathcal{S}} - \sum_a p_a S(\mathcal{C}_{a|A}(\sigma_{\mathcal{AS}})), \tag{6.9}$$

which explicitly states the link between the information related to an unread measurement, as signalized by Φ_A, with the information collected through several reductions of the form $\mathcal{C}_{a|A}$. The main message here is that the amount of information acquired by $\mathcal{O}$ about $\mathcal{S}$ is always the same regardless of the reference frame we choose to assess it. In $\mathcal{O}$'s frame we use the notion of state collapse and compute an average information, whereas in $\mathcal{O}_{\text{ext}}$'s frame the same informational content is obtained by considering a unitary evolution plus the discard of $\mathcal{O}$. From this point of view, therefore, there is no paradox. It is clear, however, that because information flows from $\mathcal{AS}$ to $\mathcal{O}$, this observer can in no way, in one's reference frame, deals with an information-preserving dynamics. In other words, in one's perspective the entropy of $\mathcal{AS}$ always decreases.

There is another involving aspect of the measurement problem that needs attention, namely, the occurrence of individual outcomes $\mathcal{C}_{a|A}(\sigma_{\mathcal{AS}}) = A_a \otimes \sigma_{\mathcal{S}|a}$ from $\mathcal{O}$'s perspective in each run of the experiment. This is no doubt a major difficulty around the issue. To discuss this point we focus on a concrete example where the z component of spin is measured for a spin-1/2 particle in a preparation $\alpha|+\rangle + \beta|-\rangle$. In the first stage of the experiment, the spin degree of freedom gets correlated, via a Stern-Gerlach field, with the spatial coordinate z of the particle. The resulting state can be written in the form

$$|\psi_{\mathcal{S}}\rangle = \alpha|+\rangle| + \bar{z}\rangle + \beta|-\rangle| - \bar{z}\rangle \in \mathcal{H}_{\mathcal{S}}, \tag{6.10}$$

where $\langle z| \pm \bar{z}\rangle \equiv \psi(z \mp \bar{z})$ stands for a probability amplitude centered at $\pm\bar{z}$. For the role of apparatus we imagine a detection array composed of ideally tiny detectors that get visible marks (via some ionizing process) upon absorption of a particle. The i-th detector starts in a state

$$|\phi_i, \varepsilon\rangle = \int \mathrm{d}z\, \phi(z - z_i)|z\rangle|\varepsilon\rangle, \tag{6.11}$$

where $|\phi(z - z_i)|^2$ is assumed to be a very sharp normalized Gaussian distribution of width δz centered at z_i and $|\varepsilon\rangle$ is a state of energy such that $\varepsilon = e$ (excited) when a mark appears in the detector and $\varepsilon = g$ (ground) otherwise. In our model,

$$\langle\varepsilon|\varepsilon'\rangle = \delta_{\varepsilon,\varepsilon'} \tag{6.12}$$

and

$$\langle\phi_i|\phi_j\rangle = \exp\big[-(z_i - z_j)^2/(8\delta z^2)\big] \approx \delta_{z_i,z_j}, \tag{6.13}$$

meaning that any two detectors and their signs are distinguishable, which is a desirable feature of any detection system. Given the finite size of the detectors, one can consistently work with a discretized space for the particle, where $\langle z_i|z_j\rangle \approx \delta_{i,j}/\delta z$ so that

$$|\pm\bar{z}\rangle = \int \mathrm{d}z\, \psi(z)\, |z \pm \bar{z}\rangle \approx \sum_k \delta z\, \psi(z_k)\, |z_k \pm \bar{z}\rangle, \tag{6.14}$$

with $\bar{z} = n\,\delta z$ for $n \in \mathbb{Z}$. By virtue of the space discretization, one has $z_k = k\,\delta z$ and therefore $z_k \pm \bar{z} = z_{k\pm n}$. Now let $|\psi_{\mathcal{A}}\rangle = \bigotimes_i |\phi_i, g\rangle$ be the initial state of the apparatus. Our model admits that upon physical interactions one has that

$$|z_k\rangle|\psi_{\mathcal{A}}\rangle \to |z_k\rangle|1_k\rangle, \tag{6.15}$$

where we have introduced the one-excitation state

$$|1_k\rangle \equiv |\phi_k, e\rangle \bigotimes_{i\neq k} |\phi_i, g\rangle \tag{6.16}$$

with $\langle 1_k|1_{k'}\rangle \approx \delta_{k,k'}$, which means that the detector at z_k gets excited whereas all the others remain unexcited. By use of this model, the initial joint state

$$|\psi_{\mathcal{S}}\rangle|\psi_{\mathcal{A}}\rangle = \sum_k \delta z\, \psi(z_k)\Big(\alpha|+\rangle|z_{k+n}\rangle + \beta|-\rangle|z_{k-n}\rangle\Big) \bigotimes_i |\phi_i, g\rangle$$

is shown to evolve to the correlated one

$$|\psi_{\mathcal{AS}}\rangle = \sum_k \delta z\, \psi(z_k)\Big(\alpha|+\rangle|z_{k+n}\rangle|1_{k+n}\rangle + \beta|-\rangle|z_{k-n}\rangle|1_{k-n}\rangle\Big). \tag{6.17}$$

We are now in position to bring to the discussion an element that, although fundamental, is rarely appreciated. It refers to the fact that in every measurement there is at least one degree of freedom that is irremediably discarded, and this is precisely the one about which we want to obtain information. In our example, the fundamentally inaccessible degrees—in fact, that is why we couple an apparatus to get information about them—are the spin and the spatial coordinate of the particle. These degrees of freedom *must* be traced out from our theoretical description. This discard is not optional; it is mandatory and irreducible. In doing so we get the following reduced state for the apparatus:

$$\rho_{\mathcal{A}} = \sum_k \delta z\, |\psi(z_k)|^2 \left(|\alpha|^2 |1_{k+n}\rangle\langle 1_{k+n}| + |\beta|^2 |1_{k-n}\rangle\langle 1_{k-n}|\right). \tag{6.18}$$

In

$$\langle 1_i|\rho_{\mathcal{A}}|1_j\rangle = \delta z \left(|\alpha|^2|\psi(z_{j-n})|^2 + |\beta|^2|\psi(z_{j+n})|^2\right)\delta_{i,j}, \tag{6.19}$$

we see that the apparatus state is diagonal in the $\{|1_i\rangle\}$ basis. Then, as far as the observable $\Lambda = \sum_i \lambda_i |1_i\rangle\langle 1_i|$ is concerned, we can ensure via definition (3.56) that $\mathfrak{I}(\Lambda|\rho_{\mathcal{A}}) = 0$, that is, given the available state $\rho_{\mathcal{A}}$ it follows that Λ is real. At the very last stage of the measurement process, information about the apparatus is transported to the observer by photons. In fact, many distinct observers can shine the apparatus and collect their own photons. The point is that the correlations generated between the photons and the apparatus will necessarily be of a classical nature because the state (6.18) is an incoherent mixture. Since no quantum correlation is generated and the local irreality of the apparatus (and of the photons) remains null, the relation (3.59) guarantees that the reality of the apparatus is preserved during this process. This shows how many observers can get information and agree about the same already-established reality, which thus reveals itself as an objective reality. Also, because the joint state of the apparatus-photons system is correlated only classically, one admits, in light of Bell's theorem, that hidden-variable theories consistent with the hypothesis of local realism are admissible as legitimate models to explain these correlations. In particular, a classical-statistical model such as the Liouvillian theory might accomplish the task in terms of deterministic Hamiltonian trajectories in phase space. However, like QM, this model would be unable to predict individual outcomes because uncertainty (in this case deriving from subjective ignorance about the initial state of the system) would still be present. In other words, the inherent statistical character of the formalism precludes precise predictions for individual runs. Hence, given the underlying determinism of such a model, the emerging result of any run of the experiment has to be interpreted as mere information updating, rather than some reality collapse. We claim that this should also be the interpretation for the quantum collapse. The quantum formalism is irreducibly statistical because it was drawn to deal with subtle scenarios involving quantum probability amplitudes, which are associated with pure superpositions. In its statistical capacity it can also describe classical-like behaviors, such as (6.18). Just like the Liouvillian formalism, however, QM is not able to predict single outcomes and this should be perfectly fine, since this is what we expect from a theory that deals with (both fundamental and subjective) uncertainties. The final acquisition of information by the observer (who cannot include himself in the theory) is then formulated as an abrupt collapse, which should not be viewed as an actual reduction of any physical element of reality.

6.2 QUANTUM REFERENCE FRAMES AND IRREVERSIBILITY

Another fundamental point that is not often appreciated in discussions about the measurement problem concerns the notion of quantum reference frames (see, e.g., Refs. [120, 121] and references therein). In spite of their complexity [122], detectors can be minimally modeled in terms of two degrees of freedom: one related to a visible sign ({excited,ground}, as we used above, or {click,ready}) and another one related to its location in space-time. Actually, the latter defines the very structure of space-time that plays the role of reference frame. In the discussion above we used the state $|\phi_i\rangle$ for the spatial component of the i-th detector. Being very sharp in the configuration space, it presumably is very wide in the momentum space, a feature that is not expected for realistic detectors. In fact, because ordinary detectors are rigidly attached to the laboratory, each one needs to simultaneously have well-defined values of position and velocity at every instant of time, for only in this case can we trust the outcomes we read in each run of the experiment and then make sense of the whole statistics observed. Formally, the observer could describe such an essentially classical detector by admitting that it has an (effective) infinite mass, in which case the uncertainty principle $\Delta z \Delta p \geqslant \hbar/2$ would remain valid whereas Δz and $\Delta \dot{z} = \Delta p/m$ vanish simultaneously. In this sense, simultaneous elements of reality for position and velocity emerge from such an intrinsic classicality of the apparatus, which comes from the fact that it is rigidly attached to and therefore defines the reference frame. To a certain extent, we can recognize here the Bohr claim about the irreducibly classical nature of the apparatus. This is not to say, however, that the apparatus is absolutely classical in any sense. In fact, an external observer who can detect the motion of the laboratory would ascribe a finite mass to the apparatus and, as consequence, could eventually find it in superposition [120, 121], that is, with no positional element of reality.

Finally, it is opportune to further elaborated on how the notion of a fundamental irreversibility, in an informational sense, emerges in the present context. The external observer $\mathcal{O}_{\text{ext}}$, before performing any measurement, describes the joint system $\mathcal{OAS}$ in terms of a closed dynamics which, as such, preserves the total information associated with $\rho_{\mathcal{OAS}}(t)$, that is, $\Delta I_{\mathcal{OAS}} = \Delta S_{\mathcal{OAS}} = 0$. In this case, if provided with precise information about $\rho_{\mathcal{OAS}}(t)$ and about the interactions among the parts, $\mathcal{O}_{\text{ext}}$ could theoretically reverse the time evolution of the system and thus get to know the initial state of $\mathcal{OAS}$. We propose to take this as a statement of informational reversibility. If, on the other hand, $\mathcal{O}_{\text{ext}}$ is given precise information about the interaction between the internal observer $\mathcal{O}$ and $\mathcal{AS}$ but has no access to the resulting state of $\mathcal{O}$ after the interaction (as in an unrevealed measurement protocol), then the initial ignorance $S_i = S(\sigma_{\mathcal{AS}})$ that $\mathcal{O}_{\text{ext}}$ has about $\mathcal{AS}$ evolves to

$$S_f = S(\Phi_A(\sigma_{\mathcal{AS}})), \tag{6.20}$$

which means that $\Delta S \geqslant 0$ and $\Delta I \leqslant 0$. Clearly, the lack of information about $\mathcal{O}$'s state (discard) implies an irreversible decrease of information. In fact, if provided with precise information about the final state $\Phi_A(\sigma_{AS})$ and the interactions between $\mathcal{A}$ and $\mathcal{S}$, $\mathcal{O}_{\text{ext}}$ would *not* be able to predict the initial state $\sigma_{\mathcal{AS}}$. With regard to the internal observer $\mathcal{O}$, who does not include oneself in the physical description, the information is not preserved as well. The initial ignorance that $\mathcal{O}$ has about the system is given by

$$\bar{S}_i \equiv S(\sigma_{\mathcal{AS}}) = S_{\mathcal{S}|\mathcal{A}} + S_{\mathcal{A}} = S_{\mathcal{A}|\mathcal{S}} + S_{\mathcal{S}}. \tag{6.21}$$

After many runs of the experiment, the average ignorance about the system $\mathcal{AS}$ is given by

$$\bar{S}_f = \bar{S}_{\mathcal{S}|\mathcal{A}} + \bar{S}_{\mathcal{A}}, \tag{6.22}$$

where $\bar{S}_{\mathcal{S}|\mathcal{A}} = \sum_a p_a S(\sigma_{\mathcal{S}|a})$ and $\bar{S}_{\mathcal{A}} = \sum_a p_a S(A_a) = 0$. Since we have

$$\bar{S}_i \geqslant S_{\mathcal{S}} = S(\sigma_{\mathcal{S}}) \tag{6.23}$$

and, via concavity,

$$S(\sigma_{\mathcal{S}}) = S(\sum_a p_a \sigma_{\mathcal{S}|a}) \geqslant \sum_a p_a S(\sigma_{\mathcal{S}|a}), \tag{6.24}$$

it follows that $\bar{S}_i \geqslant \bar{S}_f$. Hence, $\Delta\bar{S} \leqslant 0$ and $\Delta\bar{I} \geqslant 0$. Here the collapse implies gain of information, but this is also an irreversible process because $\mathcal{O}$ does not describe one's interaction with $\mathcal{AS}$ and therefore cannot reverse the time evolution to obtain information about $\sigma_{\mathcal{AS}}$. It is instructive to note that the information increase for $\mathcal{O}$, in contrast with the discard-induced information decrease for $\mathcal{O}_{\text{ext}}$, derives from the fact that $\mathcal{O}$ has access to the sequence $\{a\}$ of outcomes for the apparatus. To see this, note that

$$\begin{aligned}\Delta S &= S(\Phi_A(\sigma_{\mathcal{AS}})) - S(\sigma_{\mathcal{AS}}) \\ &= S(\Phi_A(\sigma_{\mathcal{A}})) + \sum_a p_a S(\sigma_{\mathcal{S}|a}) - S(\sigma_{\mathcal{AS}}) \\ &= H(\{p_a\}) + \Delta\bar{S}, \end{aligned} \tag{6.25}$$

where $H(\{p_a\})$ is the Shannon entropy associated with the distribution p_a. It follows that $\Delta\bar{I} = H(\{p_a\}) + \Delta I$, which proves the point. Notice that throughout this thesis we have taken the von Neumann entropy S purely as an ignorance quantifier, as in any informational framework. However, the Landauer erasure principle [58, 123], which tells us that information has effective thermodynamical implications, along with recent developments in the emerging field of quantum thermodynamics [124, 125], provides substantial license for one to conceptually connect S with the thermodynamical entropy. In this case, the apparently separated notions of informational (ir)reversibility, which we have assessed so far, and the usual one of thermodynamic (ir)reversibility may coalesce into a single concept. The take-away message here is as follows: It is the inevitable discard of degrees of freedom associated with the internal observer $\mathcal{O}$, which receive part of the information flow, that yields the fundamental informational irreversibility perceived by this observer. The external one, who deals with a closed system and no discard of information, has at hand a reversible dynamics. A central point to note is that the final acquisition of information by the internal observer (who cannot include himself in the theory) is then formulated as an abrupt collapse, which should not be viewed as an actual reduction of any physical element of reality. In what follows, we apply these ideas to discuss the physical reality in the context of Wigner's friend paradox.

6.3 WIGNER'S FRIEND PARADOX

The measurement problem in QM has raised a number of reasons for the return of deep discussions that mankind has been facing for a long time, involving the spirit of Descartes's "Cogito ergo sum", which recognizes thought, that is, mind as an element fundamental for defining reality. Such conundrums were intensely debated in the context of the so-called Wigner's friend paradox, the paradox of Schrodinger's cat, and many other puzzles inspired by these two paradoxes. The central point to note in those discussions is that Wigner's friend or Schrodinger's cat paradox cannot be completely solved without going deeply into a more fundamental question: What is the role of consciousness in the measurement process in QM? The idea that a conscious being might overlap between two mutually exclusive states as a quantum state or else the idea that a cat might be alive and dead at the same time makes clear the importance of addressing these issues. For example, for some like Wigner [127], a being with consciousness must have a different role in QM than the inanimate measuring device to conceive a resolution of the paradox. On the other hand, there are others who do not agree with such a special rule for observers in dealing with the measurement problem. Moreover, there is also the possibility, as defended by Penrose, that consciousness itself may be the result of a measurement process in a more fundamental view of the problem [6]. In the following, we discuss the most basic version of Wigner's friend paradox with the intention of using the results obtained in this thesis to formulate our own opinion on the question of the role of the observers in the QM.

In the most basic version of the paradox, the story is very similar to Everett's paradox, the only difference being that the internal observer (friend of Wigner), after making the measurements, makes a classic communication with the external observer (Wigner) informing the result of the experiment. The paradox arises in the difference between the "moments" in which the wave function collapsed. The internal observer sees the wave function collapsing at time t and then at a posterior time $t+1$ he communicates the result to Wigner. While Wigner does not receive the call, he treats the system as closed and evolving without collapse. After the call, he becomes aware of which state the system has collapsed to. Have the wave function collapsed when the internal observer became aware of the result or only when the external observer became aware?

Throughout this chapter we separate the measurement process into two steps. First, the reality of the measured observable is established when it gets maximally correlated with the apparatus. At this moment, since these correlated degrees of freedom are distinguishable from each other and because we never directly access the observable we want to measure, this latter degree of freedom is traced out, so that the apparatus ends up in a mixture, which corresponds to a state of reality for the apparatus. The same reasoning applies to the measured observable. Second, the observer look at the apparatus. To this end, photons have to get classicaly correlated with the apparatus and then delivery this information to the observer. The state for the system apparatus + photons admits a local hidden-variable model according to which the reality of its constituents are already established. What happens when the photons are collected by the observer is then nothing but Bayesian updating of information. In this thesis we defend the idea that for the establishment of physical reality, it is not necessary to assume any outstanding or fundamental role of the conscious observer; the crucial mechanism is generation of quantum correlations, via physical interactions, followed by a fundamental discard. In Wigner's friend paradox, the reality of the apparatus is established within the laboratory upon the irreducible discard of the measured observable. All further steps in the process only generate classical correlations among the constituents and, therefore, information updating. The irreducible discard of the degree of freedom that one wants to measure also dissolves the Schrodinger cat paradox. Upon interaction between cat and radioactive nucleous, the system up to a normalization factor, ends up in the state $(|0\rangle|\text{dead}\rangle + |1\rangle|\text{alive}\rangle)$, where $|0\rangle$ means that the radioactive nucleous has decayed (thus killing the cat), and $|1\rangle$ if it has not decayed (so keeping the cat alive). However, the nucleous state is inaccessible; that is why we need an apparatus (the cat) to provide information about it. Tracing out the nucleous we get $(|\text{dead}\rangle\langle\text{dead}| + |\text{alive}\rangle\langle\text{alive}|)$ for the cat, which is a state of reality. That is all an observer can infer through direct measurements on the cat (for instance, via scattered photons). Since the reality of the cat is then already defined, but still remains unknown to the observer, the collection of photons by the retina of the observer just produces the Bayesian updating of information: the cat will be revealed to be either dead *or* alive.

In what follows, we also use our approach to reinterpret the famous wave-particle duality in QM.

6.4 WAVE-PARTICLE DUALITY

Although Bohr's complementarity, proposed to account for Louis de Broglie's proposal of duality, is widely accepted as a inherent aspect of the dual manifestation of the microscopic world, there have been some dispute about its precise formulation and interpretation [54]. For Bohr, quantum systems will exhibit wave-like or particle-like behavior depending on what our experiment is supposed to measured, thus being aspects exclusively complementary. The famous double-slit experiment well illustrates the principle of wave-particle complementarity: if we do not have access to information about which path a particle ($\mathcal{P}$) crossed, the particle, behaving like a wave, passes simultaneously through both slits (according to some interpretations). Suppose now that one can get to know the pathway of each individual particle passing through the double-slit setup by observing the kickback (implied by momentum conservation) imparted onto the first slit which is used to diffract the particle wave. That is, suppose that we have a very light plate $\mathcal{S}_1$ with a single slit, which is free to move, and another plate $\mathcal{S}_2$ with two slits, which is rigidly fixed to the laboratory, as is schematically represented in Fig. 6.1.

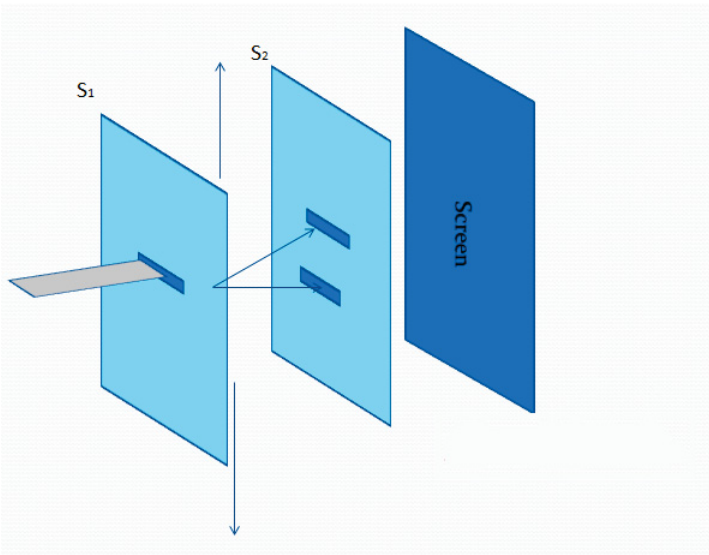

Figure 6.1: The plate is fixed in the laboratory. After the particle interacted with the light slit floating $\mathcal{S}_1$, the particle $\mathcal{P}$ moves toward a double-slit plate $\mathcal{S}_2$ which is rigidly fixed in the laboratory.

If the momentum transference between the quantum system $\mathcal{P}$ and the plate $\mathcal{S}_1$ is enough to cause a recoil in the latter in such a way that this can encode information about which slit in $\mathcal{S}_2$ the particle will pass through, then entanglement is generated between $\mathcal{S}_1$ and $\mathcal{P}$ that will be enough to destroy the interference pattern. This thought experiment has recently been conducted with "appropriate slits" (molecules) [128].

More precisely, the momentum conservation implies that if $\mathcal{P}$ moves to upwards (downwards), $\mathcal{S}_1$ has to move downwards (upwards), thus informing through which of the two slits the particle will pass in $\mathcal{S}_2$. If m and M represent the mass of $\mathcal{P}$ e $\mathcal{S}_1$, then the correlation generated in this experiment can be described by the following state [53]:

$$|\Psi_{\mathcal{P}\mathcal{S}_1}\rangle = \frac{1}{\sqrt{2}}\left(|v\rangle \otimes |-\frac{mv}{M}\rangle + |-v\rangle \otimes |+\frac{mv}{M}\rangle\right), \tag{6.26}$$

where $\pm v$ and $\pm\frac{mv}{M}$ are the respective velocities that $\mathcal{P}$ and $\mathcal{S}_1$ can assume (which are, for simplicity, considered as discrete variables).

Here we defend a framework that allows us to interpret complementarity in terms of correlations between the system and an informer (in the above case, $\mathcal{S}_1$). Our proposal offers formal definition and operational interpretation for the dual behavior of quantum systems by making links that offer a generalized information-based trade-off with reality. In fact, if we take the tools developed in this thesis, we can interpret the previously discussed experiment in the following way. Let $\mathcal{S}_1$ be our ancillary system, in such a way that it interacts with the quantum system and produces a monitoring of its momentum. Assume that after the interaction the state can be written as in 6.26. Tracing the ancila out gives

$$\rho'_{\mathcal{P}} = Tr_{\mathcal{S}_1}[|\Psi_{\mathcal{P}\mathcal{S}_1}\rangle\langle\Psi_{\mathcal{P}\mathcal{S}_1}|] \tag{6.27}$$

$$= \frac{1}{2}(|v\rangle\langle v| + |-v\rangle\langle -v|) + \frac{x}{2}(|v\rangle\langle -v| + |-v\rangle\langle v|) \tag{6.28}$$

where $x = \left|\left\langle \frac{mv}{M} \middle| \frac{-mv}{M}\right\rangle\right|$. If we rewrite $x = 1-\epsilon$, note that the state $\rho_{\mathcal{P}}$ of the particle for the case of interaction with an infinite-mass plate $\mathcal{S}_1$ would be

$$\rho_{\mathcal{P}} = \frac{1}{2}(|v\rangle\langle v| + |-v\rangle\langle -v| + |v\rangle\langle -v| + |-v\rangle\langle v|), \tag{6.29}$$

and observe that $\Phi_V(\rho_{\mathcal{P}}) = \frac{1}{2}(|v\rangle\langle v| + |-v\rangle\langle -v|)$, then we may write

$$\begin{aligned}\rho'_{\mathcal{P}} =& Tr_{\mathcal{S}_1}[|\Psi_{\mathcal{P}\mathcal{S}_1}\rangle\langle\Psi_{\mathcal{P}\mathcal{S}_1}|] && (6.30)\\ =&\rho_{\mathcal{P}} - \frac{\epsilon}{2}(|v\rangle\langle -v| + |-v\rangle\langle +v|) && (6.31)\\ =&(1-\epsilon)\rho_{\mathcal{P}} + \epsilon\Phi_V(\rho_{\mathcal{P}}) = \mathcal{M}^{\epsilon}_V(\rho_{\mathcal{P}}).\end{aligned}$$

It can be verified that $\mathfrak{I}(V|\rho_{\mathcal{P}}) = 0$ only if $\epsilon = 1$, this implies that the velocity will be real only if the system $\mathcal{S}_1$ is able to unambiguously detect the path traveled by $\mathcal{P}$. Also it follows that the irreality $\mathfrak{I}(V|\rho_{\mathcal{P}})$ is a monotonically decreasing function of ϵ since we can write the change in the degree of irreality $\mathfrak{I}(V|\rho_{\mathcal{P}})$ as

$$\Delta\mathfrak{R}(V) = S(\mathcal{M}^{\epsilon}_V(\rho_{\mathcal{P}})) - S(\rho_{\mathcal{P}}) \geq 0. \tag{6.32}$$

Note that reality of V can be clearly adjusted through the ratio between the two masses, $\frac{m}{M}$, which is a number set by the experimentalist. This is in line with Bohr's view in [101]. If $m \ll M$ then $\epsilon \to 0$, the conservation of momentum will not be able to detect the path traveled by the particle. Alternatively, we can think of a scenario where a weak measurement is conducted which is not able to extract much information about the velocity. As a consequence, this quantity remains indefinite (unreal), the interference patterns shows up, and wave-like behavior is diagnosed. On the other hand, as seen by the complementarity relation developed in this work,

$$\Delta\left(I_{\mathcal{P}:\mathcal{S}_1} + I_{\mathcal{S}_1}\right) + \Delta\mathfrak{I}(V) = 0, \tag{6.33}$$

$$\Delta I_{\mathcal{P}} + \Delta\mathfrak{R}(V) = 0. \tag{6.34}$$

In the case under inspection we also have $\Delta\mathfrak{R}(V) = E$. Now we can tell the following story about the experiment. When the momentum of the particle is monitored, entanglement is produced, which implies an increase in mutual information that leads to decreasing the available information. Variations in both the local information $I_{\mathcal{S}_1}$ associated with the subsystem $\mathcal{S}_1$ and the information shared by $\mathcal{P}$ and $\mathcal{S}_1$ directly imply variations in the irreality of V. With that, we see as a result an increase in the reality of the physical property that is being monitored, hence corpuscular behavior follows. We reinforce the idea according to which coherence is suppressed in the presence of an informer, an extra degree of freedom which, being correlated with the quantum system of interest, can detect which-path information and makes the system behave as a particle.

Our approach relies on primitive elements only, such as deterministic evolution (Schrödinger's equation), physical causation (weak and strong interactions), correlations , and the role of the informer with a partial trace, for the diagnostic of subsystems [54]. It is important to note that the partial discard plays a fundamental role since this point makes evident the contextuality of QM. The central point is that the wave behavior is expected for isolated systems or very weakly interacting ones, while the corpuscular occurs through the establishment of quantum correlations followed by discarding (fundamental or not). With that, the wave-particle duality, treated today as a principle of quantum theory, to which both radiation and matter are submitted, can in fact be abandoned!

7 CONCLUDING REMARKS

Whenever we speak of measurement, it is immediate to think of a *relation* between two entities, the measured system and the measuring device. The idea of "relation", in turn, passes through the concept of an interacting dynamics, since initially the systems are independent and then become afterwards *correlated*. The notion of dynamics finds a well-defined place within the laws of physics, since the basic interactions in nature are well described by some potential. QM, on the other hand, teaches us that the classical deterministic notion of an objective reality calls for a critical review. In this work we employed a recently developed measure of reality [53] and traditional tools of quantum information theory to get some insight into the issue. Careful experimental inspections of microscopic systems, conducted in many laboratories in the world and in several ages, have pointed out that there are many instances where the physical reality seems to be in suspension, that is, physical quantities do not have well-defined values [36]. As we have shown here, this can be achieved, e.g., by letting a particle interact with massive structures, for in such cases the (apparent frustration of) conservation laws prevent the generation of entanglement and enhance the irreality of a particle's degrees of freedom. Irreality can also be created for a given observable by means of revealed measurements of an incompatible observable. On the other hand, we also showed that any attempt to probe nature, even via arbitrarily tiny monitorings (unrevealed collapse or entanglement plus discard), leads to the emergence of elements of reality. As formally stated in the complementarity relation (4.88), the flow of quantum information from the system to the apparatus increases the reality of the monitored observable.

In the connection with measurements and quantum correlations, we have shown that if we take a distance-based formulation as a primitive notion for quantum discord, as pondered in Refs. [72, 84], then no surprise is found when replacing projective measurements with weak ones. In particular, no "super" quantum discord emerges. Rather, we find a quantifier—the weak quantum discord—that interpolates between the regime of "no quantum correlations destroyed" (when no measurement is conducted, that is, $\epsilon \to 0$)

and the regime of "all quantum correlations destroyed" (when a projective measurement is conducted, that is, $\epsilon \to 1$), in which case the quantum discord is recovered. This allows us to interpret the weak quantum discord as a measure of the amount of quantum correlations that is removed via local weak measurements. In addition, we have shown how to properly define a symmetrical weak quantum discord and briefly discussed notions of hierarchy and ordering among various discord-like quantifiers [64].

In a detailed account of the measurement process, we found another facet of this story: Information associated with the apparatus flows to the degree of freedom that we want to measure, the one that is invariably discarded. It follows that the degrees of freedom of the apparatus, in particular those that define the very space-time structure of the reference frame, become real. At this stage, QM predicts a fully incoherent mixture for the apparatus, meaning that only subjective ignorance persists about an already established reality. The final (irreversible) flow of information, which is mediated by photons that inform the observer about the state of the apparatus, materialize the information updating of the observer, a step that is out of reach of any statistical theory. In QM, this dynamically indescribable transition is called collapse [36].

It is worth emphasizing that the adoption of BA's notion of reality allows us to formalize a complementarity relation between reality and information. We find in this framework that QM predicts no objective reality for isolated systems. Elements of reality can emerge for a given observable only through the codification of information about this quantity. This process, however, does not demand the existence of a brain-endowed system to collect and interpret the information. All that is fundamentally necessary is the presence of physical degrees of freedom that can get correlated with the observable and thus encode information about it. The information that flows to these degrees of freedom makes the reality emerge and become potentially accessible to brain-endowed observers [36].

We consider that the enigmas of wave-particle duality and the wave function collapse can have a satisfactory explanation by using the tools developed in this thesis. Hopefully, this may help to demystify the quantum strangeness which is so confusing to scientists and even more so to the general public.

As a future research, we aim at using the tools developed in this work to propose an axiomatic theory of measurement in physical theories with possible emphasis on correlations, information, thermodynamics, relativity (quantum references frames). Another point we want to explore would be to bring to that discussion the notion of information gain in the measurement process from a quantum thermodynamics point of view. The connection with thermodynamics is not trivial because von Neumann's entropy is not always recognized as synonymous with traditional thermodynamics entropy. This only occurs when the system is in equilibrium, and its state is the traditional Gibbs state. The delicate issue is precisely this: the goal of thermodynamics as we know it is to describe processes between two equilibrium situations, which involve or are defined by reservoirs. In generic quantum dynamics, on the other side, we do not necessarily have states of equilibrium.

Another rather fundamental is to look at elements of reality from the point of view of distinct quantum reference frames. How does the reality of physical observables change when one move from one to another quantum reference frame? As far as quantumness notions such as entanglement, discord, coherence, irreality, and the like, are concerned, it there some invariant quantity upon changes of quantum reference frames? We believe that these questions define another fascinating quest through the foundational substratum of QM.

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